24414

JOURNAL

OF THE

DEPARTMENT OF SCIENCE

University of Calcutta

Journal

of the

Department of Science

Vol. V

CALCUTTA UNIVERSITY PRESS
1923

PRINTED BY BHUPENDRA LAL BANERJEE
AT THE CALCUTTA UNIVERSITY PRESS, SENATE HOUSE, CALCUTTA

Reg. No. 19B, July, 1923—500.

CONTENTS

Chemistry

Physics

Mathematics

Botany

Geology

CHEMISTRY

Chloropicrin as a Reagent for the Diagnosis of Mercaptans and Potential Mercaptans.

BY

Sir Praphulla Chandra Ray and Radhakishen Das.

The reactivity of chloropicrin towards the mercaptans has formed the subject of a previous communication (T. 1919, *115*, 1308). It has been shown that with the typical mercaptans the reaction proceeds as follows :—

$$3RSH + NO_2CCl_3 = (RS)_3CNO_2 + 3HCl.$$

At the ordinary temperatures; but at higher temperatures the reaction is of the same order, only nitrous fumes are given off from two molecules of the condensed product, thus :—

$$\geq C - NO_2 + O_2NC \leq \; = \; > C - O - C < \cdot + N_2O_3.$$

In the present investigation, mercaptans of divers types have been similarly treated and it has invariably been found that there is no deviation whatever from the reaction given above.

When, however, potential or imino-mercaptans, *e.g.*, thiocarbamide and its alkyl and arylated derivatives, thioacetanilide, thioacetamide, etc., are substituted in place of real mercaptans, the reaction still takes place, but with a marked difference: Sulphur being invariably separated and in some cases, hydrogen sulphide evolved. The reactions under this head, may, however, be grouped under two classes :—

1. Those in which there is complete separation of sulphur, *e.g.*, thioacetanilide and chloropicrin. The reaction proceeds according to the following scheme :—

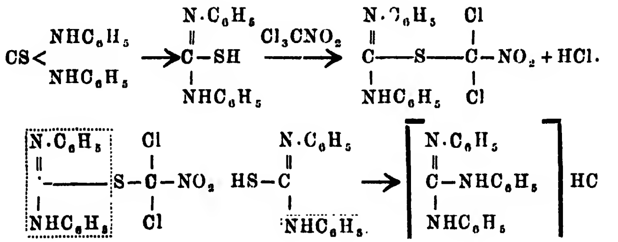
$$\begin{matrix} PhNHCSMe \\ Ph\boxed{NHC}\boxed{S}\boxed{Me} \end{matrix} + Cl_3CNO_2$$

$$\longrightarrow [Me \cdot C(:N \cdot Ph) - C \cdot Cl : CMe \cdot NH_2], \; HCl.$$

During the reaction copious evolution of hydrogen sulphide takes place with the separation of sulphur and finally of nitrous fumes.

The imino-group of one molecule is also reduced to amino. That the compound has one of its chlorine atoms disposed of as above and is a hydrochloride is proved by the fact that it is soluble in water and has its chlorine partially thrown down by silver nitrate. The presence of an amino as also of a phenyl-amino-group evidently endows it with basic property.

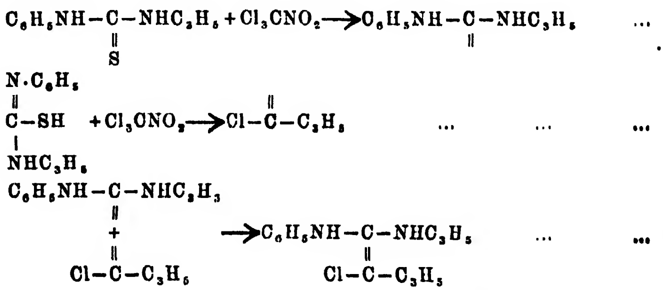
In the case of diaryl substituted thiocarbamides, although there is complete separation of sulphur, the resulting product is a tri-substituted guanidine derivative. The course of the reaction is as follows :—



... (1)

As guanidine is a strong base, the hydrogen chloride which is eliminated as a result of the first reaction naturally combines with the base with the formation of the hydrochloride.

When, however, one of the substituents is an alkyl-group and the other an aryl-group, the reaction takes a somewhat different turn. There is complete separation of sulphur as in the above instances but no guanidine derivative is formed. Thus :—



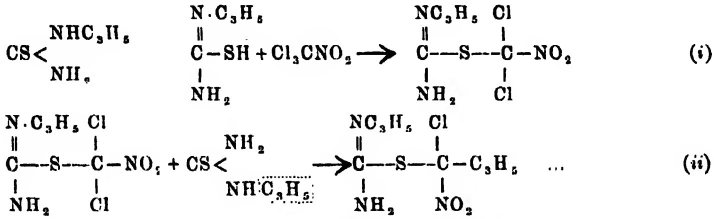
... (*i*)

... (*ii*)

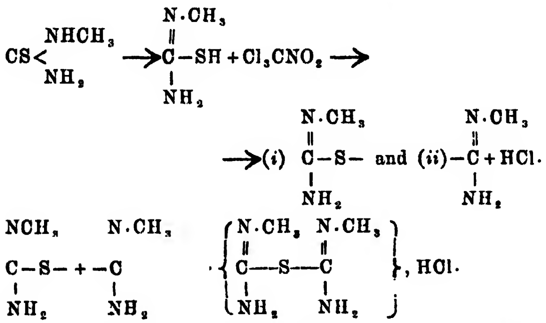
... (*iii*)

Here the reaction is evidently bimolecular. Sulphur separates out from one molecule; the other molecule is ruptured and its residue (in this instance, the allyl-group) attaches itself to the chloropicrin residue. The bivalent group thus formed takes the place of the detached sulphur atom giving rise to the compound as indicated above.

2. Those in which two or three molecules of the thio-body simultaneously take part in the reaction, of which one or two molecules remain intact and simply behave as imino-mercaptans (by tautomerisation), whilst the second or the third molecule, as the case may be, is acted upon by one or more atoms of chlorine yielded by the chloropicrin, which in the process is disintegrated; and a portion of the ruptured molecule which is stable under the conditions, takes the place of the liberated chlorine atoms. Thus:—



Methyl-benzyl-thiourea, Allyl-benzyl-thiourea come under the above category. Mono-methyl-thiourea, however, deviates somewhat from the above reaction. Here two molecules of the parent substance undergo rupture and the two distinct residues of each attach themselves to form the compound as shown below:



This is perhaps a singular case as no residue of chloropicrin itself takes part in the formation of the compound.

As the resulting compound contains 4 amino and imino-groups it naturally behaves as a strong base and the liberated hydrogen chloride of the first reaction combines with it forming a salt (a hydrochloride) and this is proved by the fact that its aqueous solution forms silver chloride with silver nitrate.

The reaction between p-chlorophenylthiourea and chloropicrin is also of the same nature. Here two molecules take part in the reaction, one of which acts like an ordinary mercaptan (by tautomerisation)

and the other is broken up and its degradation product, *viz.*, (:C: NH) attaches itself to the nucleus of chloropicrin. Thus:—

$$\begin{array}{l} NH_2CSNHC_6H_4Cl \longrightarrow NH{:}C(NHC_6H_4Cl)SH \longrightarrow NH{:}\underset{\displaystyle S}{\overset{|}{C}}-NHC_6H_4-\underset{\displaystyle CCl_2.}{\overset{|}{C}}{:}NH \end{array}$$

In this reaction nitrous fumes are evolved, *i.e.*, the nitro-group is detached.

That the chlorine atom of the chlorophenyl group parts company is proved by an independent reaction. When this thio-carbamide is treated with mercuric nitrite, a nitrite free from chlorine of the empirical formula $C_7H_6O_5N_4S_2Hg_3$ is obtained, which may be graphically represented as

$$NO_2Hg-S-C_6H_4-NH-C({:}NH)-S({:}HgO).\ HgNO_2.$$

Thioacetamide as also thiobenzamide present some peculiarities. Here three molecules simultaneously take part in the reaction; of which two molecules remain intact, while the third only has its sulphur atom detached.

$$3CH_3CSNH_2 \longrightarrow 3CH_3C({:}NH)-SH \longrightarrow \{CH_3C({:}NH)-S\}_2{=}\underset{\displaystyle C({:}NH)\cdot CH_3.}{\overset{}{\underset{|}{C}}}-NO_2$$

It will be seen the two atoms of chlorine of chloropicrin act in the manner indicated above and at the same time, another portion of the reagent acts drastically upon the thio-body breaking up its molecular structure and the complex $CH_3C({:}NH)$ attaches itself to the chloropicrin nucleus. By a series of secondary reactions sulphur is liberated and hydrogen sulphide is formed; the latter in turn, reduces the nitro-group into NH_2-group and the final product is:—

$$\{CH_3C({:}NH)-S\}_2{=}\underset{\displaystyle C({:}NH)\cdot CH_3.}{\underset{|}{C}}-NH_2$$

The evidence afforded by ethylene-pseudo-thiourea is very interesting. This body may be regarded as derived from thioline-μmercaptan $\begin{array}{l}CH_2-S\\ |\qquad\quad \geq C\cdot SH\\ CH_2-N\end{array}$ by the substitution of S : by :NH; but while the latter behaves like a real mercaptan (as in Expt. 3), the former would seem to belong to the ketonic variety, *i.e.*, it is a potential mercaptan. The reaction takes place as follows:—

$$C_2H_4{<}\begin{array}{l}NH\\ \\ N\end{array}{\geq}C\cdot SH \longrightarrow C_2H_4{<}\begin{array}{l}NH\\ \\ N\end{array}{\geq}CS-\underset{\displaystyle NH_2}{\overset{\displaystyle Cl}{C}}-N{<}\begin{array}{l}CH_2\\ |\\ CH_2.\end{array}$$

The residue of $C_2H_4{<}\begin{array}{l}NH\\ \\ N\end{array}{\geq}CSH$ or $(C_2H_4 : N\text{-})$ from the ruptured

molecule, attaches itself to the chloropicrin nucleus. Here also sulphur separates out and hydrogen sulphide is formed and the NO_2-group is reduced to NH_2-group.

EXPERIMENTAL.

1. *Potassium salt of* 2-*thiol*-5-*thio*-4 *phenyl* 4 : 5-*dihydro*-1 : 3 : 4-*thiodiazole and chloropicrin.*

The reaction at 50° has already been described (T. 1919, 115, 1311); when it is carried on at 100°, nitrous fumes are evolved and by the condensation of two molecules of R_3-C-NO_2 (l. c. 1131), the compound R_3-C-O-C-R_3 was obtained along with the previous one of m.p. 129°. On treating the mixture with boiling methyl alcohol the oxy-compound was dissolved out. It was purified by recrystallisation, m.p. 106-107°.

0·0245 gave 00·075$BaSO_4$ S=41·74.

0·0381 gave 0·0601 CO_2 and O. 0127 H_2O C=43·02 H=3·67
$C_{50}H_{30}ON_{12}S_{18}$ requires S=41·44; C=43·17 ; H=2·16.

2. *Potassium salt of* 2 : 5-*dithiol*-1 : 3 : 4-*thiodiazole and chloropicrin.*

The reaction at 100° has already been described.

At the room temperature (30°), the reaction does not proceed so far as to evolve nitrous fumes to give the oxycompound. The product was purified by crystallisation from alcohol, m.p. 152° sharp.

0·1265 gave 0·0786 CO_2		C=16·98
0·1209 „ 21·6 c. c. N_2 at 32° & 760 m.m.		N=19·52
0·0764 „ 0·2631 $BaSO_4$		S=51·34
$C_8O_4N_8S_9$ requires C=17·14 ;	N=19·99 ;	S=51·43

3. 2-*mercapto*-4 : 5-*dihydro-thiazole and chloropicrin.*

The parent substances in alcoholic solution were refluxed for four hours. On cooling, no separation of sulphur took place. It was filtered and concentrated when a crystalline mass was obtained which was filtered. The product was recrystallised from alcohol. The white crystals had the m.p. 95° sharp.

0·1003 gave 12·0 c. c. N_2 at 30° & 759 m.m.	N=13·21
0·0902 „ 0·3078 $BaSO_4$	S=46·84
$C_{10}H_{12}O_2N_4S_6$ requires N=13·59 ;	S=46·50

4. *Potassium salt of ethylene thio-glycol and chloropicrin.*

The potassium salt was dissolved in alcohol to which chloropicrin was added drop by drop, at the ordinary temperature. A violent reaction ensued and enormous rise in temperature took place. A yellow mass began to be thrown down. Copious evolution of nitrous fumes also took place. It was then refluxed as usual in order to complete the reaction. On cooling, the yellow mass was filtered off from the mother liquor and washed with alcohol and dried. It was next triturated with water and filtered. The aqueous filtrate was evaporated to dryness and a portion was ignited on a platinum spatula. The residue after ignition was dissolved in water and found to be a chloride (potassium) by the silver nitrate test. After drying the yellow mass was crystallised from hot nitro-benzene in yellow powder. It shrinks at 120° and melts at 123°. In this case free sulphur was proved to be absent.

In another experiment, the reaction was allowed to take place at a lower temperature, *viz.*, that of ice. In this case also, nitrous fumes escaped and the resulting product is the same as above.

0·0647 gave 0·3089 $BaSO_4$ S.[1] =65·57

0·2088 „ 0·2117 CO_2 & 0·0904 H_2O; C=28·14; H=4·80

$$\left\{C_2H_4<\begin{matrix}S-\\SH\end{matrix}\right\}_3\equiv C-O-C\equiv\left\{C_2H_4<\begin{matrix}S-\\SH\end{matrix}\right\}_3$$ requires S=64, 21; C=28.09 H=5.02.

5. *Pinacollylsulphocarbamide and chloropicrin.*

The parent substances in alcoholic solution were refluxed for four hours. No separation of sulphur took place. The reaction mixture, after filtration and subsequent evaporation, yielded an oil. The oil was dissolved in alcohol and precipitated by ether. It did not solidify. A brown oil was obtained.

0·1217 gave 0·1919 CO_2 C=43·01.

0·0519 „ 0·0627 $BaSO_4$ & 0·0117 AgCl S=16·59; Cl=8·91.

requires C=44 10; S=15·80; Cl=8·71.

[1] Sulphur was estimated by fusion with potassium nitrate and sodium carbonate and hence a slightly high percentage of sulphur is sometimes yielded.

Ethylmercaptan and chloropicrin :—

The components in alcoholic solution did not react with each other when heated under reflux, nor even when heated in a sealed tube at 100°. But on raising the temperature to 220-240°, it was found that a minute quantity of a white substance was formed, which was not soluble in carbon bisulphide, proving that no sulphur was eliminated. As no pure product could be obtained from the reaction mixture, no definite conclusion could be assigned to it.

POTENTIAL MERCAPTANS.

6. *Para-chlorophenylthiourea and chloropicrin.*

The reacting substances in alcoholic solution were refluxed as usual for five hours. At first the thiocompound went into solution. Nitrous fumes are evolved and a quantity of sulphur separated out. The mother liquor, on concentration, gave white crystals which were washed with carbonbisulphide. It was then recrystallised from alcohol ; m. p. 245-246°.

0·0801 gave 0.1243 CO_2 C=42·33

0·1056 gave 0.1205 AgCl & 0.0915 $BaSO_4$ Cl=28·22; S=11.90

$C_9H_7N_3Cl_2S$ requires C=41·53; Cl=27·31; S=12·31

The corresponding compound with mercuric nitrite had the formula $C_7H_6O_5N_4S_2Hg_3$

Found—Hg=57·57 ;	S=4·64 ;	C=22·27 ;	H=1·94.
Calc.— Hg=58·07 ;	S=4·64 ;	C=22·64 ;	H=1·60.

7. *Mono-methylthiourea and chloropicrin.*

The components in alcoholic solution refluxed as usual for five hours. As the reaction went on, sulphur began to be deposited, which was filtered off. The alcoholic mother liquor was concentrated to drive off the excess of chloropicrin. It was washed with carbonbisulphide to free it from the minute trace of sulphur which contaminated the product; it was dissolved in alcohol and precipitated by ether as an oil. On keeping it in a vacuum desiccator, it became converted into a semisolid mass. It was very difficult to get the product in a crystalline form. After repeated trials and failures, the

following method was found to be satisfactory. The semisolid mass was treated with acetone in a beaker under constant teasing to which 3 or 4 drops of methyl alcohol added; when, all on a sudden, a white crystalline substance was obtained. Its m. p. was not sharp and so it was dissolved in a small quantity of methyl alcohol and fractionally precipitated by acetone in the form of white shining crystals. m. p. 222°. That it is a hydrochloride is proved by the fact that silver nitrate precipitates silver chloride from its aqueous solution.

0·1376 gave 0·1288 CO_2 & 0. 0830 H_2O C=25·53; H=6·70
0·1166 gave 0·0900 AgCl &0·1459 $BaSO_4$ Cl=19·10; S=17·19
0·0548 gave 15·20 c.c. N_2at 34° & 759m.m. N=29·99
$C_4H_{11}N_4SCl$ requires C=26·3; Cl=19·45; S=17·54; N=29.90

8. Mono-allylthiourea and chloropicrin.

The reacting substances in alcoholic solution refluxed as usual. Sulphur began to be deposited which was filtered off. The alcoholic mother liquor, on concentration, gave a brown oil which was insoluble in ether, whereas the parent substance (allylthiourea) is soluble in ether. The oil was dissolved in acetone and precipitated by ether. It was further purified by repeating the above process. A golden oil was thus obtained.

0·1238 gave 0·1769 CO_2 C=38.39
0·2821 gave 0·2586 $BaSO_4$ S=12.58
0·1421 gave 0·0792 AgCl Cl=13.79
$C_8H_{12}N_3SO_2Cl$ requires C=38.48; S=12.80; Cl=14.20

9. Allyl-benzylthiourea and chloropicrin.

The parent substances in alcoholic solution were refluxed as usual for five hours. The solution assumed a dark brown colour which afterwards became light yellow and sulphur was separated out. The process of purification was the same as in the previous case.

0·1212 gave 0·2548 CO_2 & 0.0754 H_2O C=57·33; H=6·91
0·1552 gave 0·0605 AgCl Cl=9·64
0·1452 gave 0·0858 $BaSO_4$ S=8·11
$C_{18}H_{20}N_3O_2SCl$ requires C=58·53; H=5·13; Cl=9·11; S=8·22

10. Methyl-benzylthiourea and chloropicrin

The method of preparation and purification is the same as in the case of monomethylthiourea and chloropicrin. m. p. 176-177°.

0·0905 gave 0·1377 CO_2 & 0·0570 H_2O C=56·56 ; H=6·20
0·0877 gave 0·0568 $BaSO_4$ S=8.81

$C_{17}H_{18}N_3O_2SCl$ requires C=56·27 ; H=4·97 ; S=8·82

11. Ethylene-pseudo-thiourea and chloropicrin.

The usual method was adopted. The resulting product was recrystallised from hot water in shining plates; m. p. 270°.

0·1414 gave 0·1803 CO_2& 0·0814 H_2O C=34·79 ; H=6·40
0·0756 gave 18·00 c.c. N_2 at 23°& 760 m.m. N=27.07
0·0992 gave 0·0720 AgCl & 0·1154 $BaSO_4$ Cl=17·95 ; S=15·98

$C_6H_{11}N_4SCl$ requires C=34·87 ; H=5·33; N=27·12 ; Cl=17·19 ; S=15·50

12. Thioacetamide and chloropicrim.

The components in alcoholic solution refluxed usual for three hours. Copious evolution of H_2S took place with the separation sf sulphur. The alcoholic mother liquor, on spontaneous evaporation, gave silky white needles, which, on further purification, melted at 103° sharp.

0·1133 gave 26·50c·c. N_2 at 32^c & 759m.m. N=25.62
0·884 gave 0·1903 $BaSO_4$ S=29·56

$C_7H_{14}N_4S_2$ requires N=25·69 ; S=29·36

13. Thiobenzamide and chloropicrin.

The reaction took place above and the method of purification was also the same. The compound had the m.p. 88° sharp.

0·1065 gave 0·2578 CO_2& 0·0447 H_2O ; C=66·01 ; H=4·60
0·0612 gave 7·40 c.c. N_2 at 32°& 759 m.m. N=13·20
0·0854 gave 0·1049 $BaSO_4$ S=16.87

$C_{11}H_{20}N_4S_2$ requires C=65.34 ; H=4.97; N=13.86 ; S=15·85.

14. *Thiocarbanilide and chloropicrin.*

The components in alcoholic solution refluxed as usual for three hours. Thiocarbanilide went into solution and the liquid assumed a yellow color and sulphur separated out. On cooling, it was filtered off and the alcoholic mother liquor, on concentration, gave a semisolid mass (brownish), the tarry matter being also present. The semi-solid mass was washed with ether to remove the tarry matter, when a white mass, slightly colored, was obtained. It was afterwards washed with acetone and recrystallised from hot water in glistening white crystals; m.p. 252°.

01415 gave 0·0640 AgCl Cl=11·18
01037 gave 0·2653 CO_2 & 0·0633 H_2O; C=69·79; H=6·75
0·0814 gave 9·60 c.c. N_2 at 27° & 763m.m. N=13·23

$C_{19}H_{18}N_2Cl$ requires Cl=10·97; C=70·47; H=5·56; N=12·98

15. *S-ditollyl-thiourea and chloropicrin.*

The method of preparation and purification was the same as in the previous case. m.p. 230°.

0·1945 gave 0·3796 AgCl Cl=10.12
0·1347 gave 0·3575 CO_2 & H·0857 H_2O C=72·38 A=7·06

$C_{22}H_{24}N_3Cl$ requires Cl=9·71; C=72 23; H=6·57

16. *Allyl-phenylthiourea and chloropicrin.*

The components as usual refluxed for four hours. Sulphur separated out. which was filtered off. The alcoholic mother liquor, on concentration, gavs a brown oil, which was washed with carbonbisulphide to remove the slight trace of sulphur and washed with ether and dissolved in alcohol and filtered and precipitated by ether when a yellow oil was obtained. The oil was dissolved in water, filtered and evaporated to dryness on the water bath. A golden yellow oil changed

into stiff mass (oil) in vacuuo was thus obtained. It was very hygroscopic.

0·0934 gave 0·2364 CO_2	C=68.55
0·1310 gave 15·0 N_2 at 27° & 763 m.m.	N=11.13
0·1023 gave 0·0604 AgCl	Cl=14.61

$C_{14}H_{17}N_2Cl$ requires C=67·59; N=11·27; Cl=14·29.

17. Thioacetanilide and chloropicrin.

The components in alcoholic solution were refluxed as usual. Sulphur was deposited and in the latter part of the reaction there was evolution of nitrous fumes. The mother liquor, on concentration, gave a semisolid mass which was dissolved in acetone and precipitated by ether. Finally, the white product was recryatallised from water and was obtained in the shape of shining plates; m.p. 186-187°.

0·0486 gave 5·50 c.c. N_2 at 29° & 759 m.m.	N=11·38
0·0685 gave 0·1357 CO_2 & 0·0405 H_2O	C=54·03; H=6·71
0·0749 gave 0·0857 AgCl	Cl=28·30

$C_{11}H_{14}N_2Cl_2$ requires N=1142; C=53·87; H=5·78;
Cl=29·00

Thiocarbamide was similarly treated with chloropicrin in alcoholic solution. There was separation of sulphur. The resulting compound was an oil which baffled all attempts at purification.

SUMMARY.

Seventeen typical thio-compounds have been subjected to investigation. Of these five are known to be *real mercaptans* and the rest as *potential ones*. In not a single instance has it been found that there is separation of sulphur in the case of real mercaptans; whereas sulphur is either partially or completely eliminated from the potential ones. Chloropicrin may, therefore, be safely used as a reagent for the diagnosis of mercaptans and potential mercaptans respectively.

Triethylene Tri-and Tetra-Sulphides and their Derivatives.

Part II.

BY

Sir Praphulla Chandra Ray.

The preparation and properties of these cyclic polysulphides have already been described (Trans. Chem. Soc. 1920, 117, 1090). Their various derivatives will form the subject of the present communication.

1. Triethylene trisulphide and mercuric nitrite: The compound conformed to the formula $(C_2H_4)_3S_3.2Hg(NO_2)_2$.

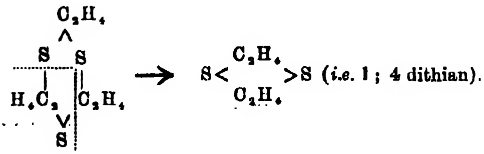
2. Triethylene trisulphide and mercuric chloride. In this case the compound had the formula $(C_2H_4)_3S_3.3HgCl_2$; evidently on account of its comparatively heavier molecular weight only two molecules of the nitrite can combine with a molecule of the polysulphide, while it can combine with the expected three molecules of the chloride. These compounds may be regarded as sulphonium derivatives in which the divalent sulphur atoms in the ring become tetravalent thus:—

$$>S + HgCl_2 \longrightarrow >S<{}^{HgCl}_{Cl}$$

When the mercuric nitrite mercaptide is treated with ethyl iodide the corresponding trisulphonium derivative, $(C_2H_4)_3S_3$, HgI_2, $2EtI$ is obtained (*cf.* Trans. Chem. Soc. 1919, *115*, 262) as also a derivative 1 : 4 Dithian $(C_2H_4)_2S_2.HgI_2.EtI$.

3. Triethylene tetrasulphide and mercuric chloride. Two distinct compounds have been obtained. The β-modification (m.p. 104°) gave one with the formula $(C_2H_4)_3S_4.\frac{1}{2}HgCl_2$, whilst the δ-modification (m.p. 59—60°) gave the compound $(C_2H_4)_3S_3.HgCl_2$. *i.e.*, in this latter case one atom of sulphur in the ring drops off. This remarkable property will be noticed below in some other instances,

e.g., in combination with ethyl iodide or platinic chloride. The underlying principle seems to be that when this complex sulphide ring is loaded with a compound having a high molecular weight, the former is put to a state of strain and has thus to part either with an atom of sulphur or the complex, $-C_2H_4-S-$; thus :



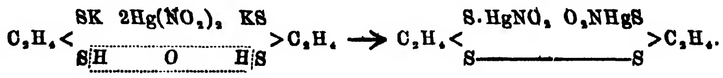
The polymerisation of these sulphides has already been referred to ; the δ-modification has been proved to be made up of four associated molecules and it was suspected that those with higher melting points, had been formed by the coalescence of more than four simple molecules (l. c. p. 1091). The β-modification yields, with ethyl iodide, a compound having the formula, $\{(C_2H_4)_3S_4\}_6$. EtI ; whilst the γ-modification yields two derivatives having the formnlae, $\{(C_2H_4)_3S_4\}_4$.EtI, and $\{(C_2H_4)_3S_4\}_5$.EtI respectively. In marked contrast with the β-modification is the behaviour of the δ-variety towards ethyl iodide; when it is refluxed with this reagent it began to evolve hydrogen sulphide. The reaction was continued for several hours till no more gas was given off. The product crystallised from hot acetone had the m. p. 77° sharp. It conformed to the formula, $C_{18}S_{14}H_{25}.C_2H_5I$, but no definite constitution could be assigned to it.

4. Triethylene disulphide di-mercaptan and mercuric nitrite. The corresponding di-nitrite mercaptide $(C_2H_4)_3S_2<\begin{matrix}S.HgNO_2\\S.HgNO_2\end{matrix}$ has been obtained, which by interaction with ethyl iodide, yielded two derivatives having identical empirical formula, *viz.*, $(C_2H_4)_4S_3$, HgI_2, 2EtI, and are instances of geometrical isomerism. Evidently they are trisulphonium compounds to which the following constitutional formula,

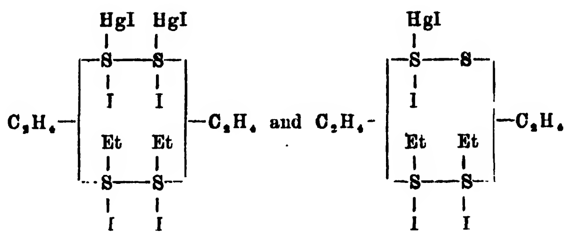
$$\begin{array}{ccccccc} & & HgI & & Et & & Et & \\ & & | & & | & & | & \\ C_2H_4 & - & S & -C_2H_4- & S & -C_2H_4- & S & -C_2H_4 \\ & & | & & | & & | & \\ & & I & & I & & I & \end{array}$$

may be assigned. (*cf.*, Trans. Chem. Soc. 1919, *115*, 262).

5. **Ethylene mercaptan (thio-glycol) and mercuric nitrite.** This interaction has already been studied (Trans. Chem. Soc. 1916, *109*, *605*). A purer product is obtained when instead of the thioglycol itself its potassium derivative $C_2H_4<^{SH}_{SK}$ is used. The mercaptide nitrite conforms to the formula $(C_2H_4)_2S_4.(HgNO_2)_2$. Evidently the following reaction takes place



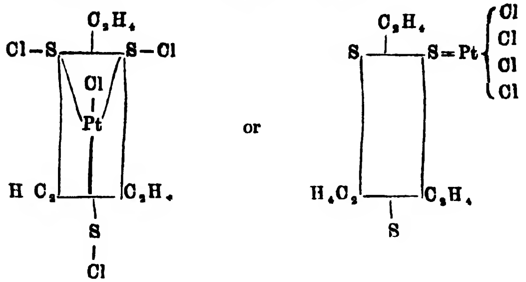
As the thioghycol is very liable to aerial oxidation the two molecules coaleace as shown above. This mercaptide nitrite, again, by interaction with ethyl iodide has been found to yield three distinct sulphonium derivatives; two conforming to the formula $(C_2H_4)_2S_4$, $2HgI_2$, $2EtI$, and the third to the formula $(C_2H_4)_2S_4$, HgI_2, $2EtI$. These may be graphically represented as



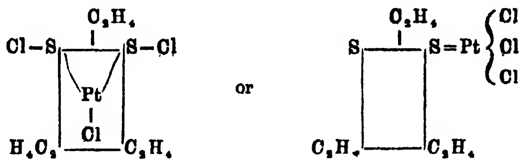
In the former all the four sulphur atoms have become tetravalent whilst in the latter one atom, however, remains as divalent. The first offers an interesting case of geometrical isomerism as both the varieties though identical composition differ widely in melting point and solubility (*vide* Experimental). The compound, $(C_2H_4)_2S_4$, HgI_2, EtI, has already been described (Trans. Chem. Soc. 1917, 111, 105) and incidentally it was stated that " whenever there are more than two atoms of sulphur in a molecule some (but not all) can become quadrivalent." It is now found that not only two or three but all the sulphur atoms in the ring can become tetravalent.

Triethylene trisulphide and platinic chloride. Two distinct chloro-mercaptides have been obtained conforming to the formula $(C_2H_4)_3S_3.PtCl_4$, and $(C_2H_4)_3S_3.PtCl_3$ respectively. In the latter

case an atom of sulphur drops off (*vide* ante, p. 2). Their constitution formulae are evidently as given below.



In the symmetrical formula all the sulphur atoms as also the platinum atoms become tetravalent, the chlorine atoms being disposed of as shown in the scheme; whereas in the unsymmetrical formula only one atom of sulphur becomes tetravalent and the platinum hexavalent, similarly the second chloro-mercaptide may be represented as follows:—



The platinum being regarded as trivalent or pentavalent.[1]

EXPERIMENTAL

Triethylene trisulphide and mercuric nitrite.

Method of preparation: To the dilute alcoholic solution of the sulphide was added in a thin stream under constant stirring a solution of sodium mercuric nitrite. A copious precipitate was obtained, which was washed and dried as usual in a vacuum desiccator. The crystals were nacreous and had a faint yellow tint; when treated with hydrochloric acid they evolved red fumes.

0·1154 gave 0·0694 Hg; 0·1207 gave 7·6 cc. N_2 at 31°C and 759 mm. pressure.

Found: Hg=51·86; N=6·91.

Calc. for $C_6H_{12}S_3Hg_2N_4O_8$; Hg=52·36; N=7·33.

[1] Further corroborative evidence will be adduced in the next paper on "The Varying Valency of Platinum with respect to Mercaptanic Radicals."

Triethylene trisulphide and mercuric chloride.

In this case alcoholic solutions of the components were added.

0·2435 gave 0·1457 Hg and 0·2129 AgCl.

Found: Hg=59·84; Cl=21·63;

Calc. for $C_6H_{12}S_3Hg_3Cl_6$; Hg=60·42; Cl=21·45.

When the mercaptide nitrite described above was refluxed with ethyl iodide for three to four hours, and the reaction-product allowed to cool, a crystalline mass was deposited. The mother-liquor was now decanted off and on addition of acetone to the impure crystals, the colouring and tarry matter was removed and needle-shaped grey crystals were obtained; m.p. 128° sharp. The acetone filtrate when mixed with about one-fourth its bulk of methyl alcohol and allowed to evaporate spontaneously. yielded leafy, almost white crystals, m.p. 86.°

The product insoluble in acetone (m.p. 128°) conformed to the formula $(C_2H_4)_3S_3.HgI_2.2EtI$.

0·1030 gave 0·0231 Hg and 0·1037 AgI

Found: Hg=22·49; I=54·35; Calc. Hg=21·14; I=53·70

The crystals deposited from the acetone-methyl alcohol mixture (m.p. 86°) conformed to the formula $(C_2H_4)_2S_2.HgI_2.EtI$.

0·3376 gave 0·0920 Hg and 0·3275 AgI

0·1689 „ 0·1066 BaSO

Found: Hg=27·28; I=52·42; S=8·67;

Calc. Hg=27·40; I=52·19; S=8·77.

Triethylene tetrasulphide and mercuric chloride.

A. From the β-modification:—Alcoholic solutions of the components were added. The compound had the formula

$$(C_2H_4)_3S_4.1\tfrac{1}{2}HgCl_2.$$

0·1451 gave 0 0703 Hg and 0·0970 AgCl

0·1364 „ 0·0564 CO_2 and 0·0264 H_2O;

0·1192 „ 0·1738 $BaSO_4$.

Found: Hg=48·45; Cl=16·54; S=20·23; C=11·28; H=2·16.

Calc. Hg=48·54; Cl=17·23; S=20·71; C=11·65 H=1·94.

B. From the δ-modification :—The compound had the formula

$(C_2H_4)_3S_3.HgCl_2$.

0·2318 gave 0·1032 Hg and 0·1566 AgCl;

0·0998 „ 0·1605 $BaSO_4$.

Found: Hg=44·53; Cl=16·74; S=22·09;

Calc. Hg=44·34; Cl=15·74; S=21·29.

Triethylene tetrasulphide and ethyl iodide.

The β-modification, when refluxed with ethyl iodide for about three hours, was turned into a thick liquid, which on cooling gave an impure solid mass, which recrystallised from hot benzene yielded white crystals, (m.p. 103°); they conformed to the formula

$\{(C_2H_4)_3S_4\}_5.C_2H_5I$.

0·0764 gave 0·0140 AgI; and 0·2936 $BaSO_4$

Found: I=9·90; S=52·78; Calc. I=10·45; S=52·64.

The γ-modification similarly treated gave a product which, when recrystallised from hot benzene, gave a crop of white crystals, m.p. 96° which conformed to the formula $\{(C_2H_4)_3S_4\}_4$.EtI.

0·0843 gave 0·0194 AgI; and 0·3103 $BaSO_4$

Found: I=12·44; S=50·56; Calc. I=12·65; S=50·99.

The mother-liquor of the above on slow and spontaneous evaporation gave a second crop which had the m.p. 70° and which conformed to the formula $\{(C_2H_4)_3S_4\}_2$.EtI.

0·0917 gave 0·0345 AgI and 0·3072 $BaSO_4$

0·1224 „ 0·1324 CO_2 and 0·0613 H_2O.

Found; I=20·33; S=43·12; C=29·50; H=5·56;

Calc. I=21·90; S=44·14; C=28·97; H=5·00.

The product of the interaction of the δ-modification and ethyl iodide.

The compound had the empirical formula $C_{18}H_{25}S_{14}.C_2H_5I$.

0·0955 gave 0·0261 AgI and 0·3650 $BaSO_4$

0·0745 „ 0·0790 CO_2.

Found: I=14·17; S=52·49; C=28·91;

Calc. I=15·03; S=53·02; C=28·40.

Triethylene disulphide dimercaptan and mercuric nitrite.

The methyl alcoholic solution of the di-mercaptan treated with mercuric nitrite, yielded the corresponding mercury nitrite mercaptide, which had the formula $(C_2H_4)_3S_4.(HgNO_2)_2.2\frac{1}{2}H_2O$.

0·3717 gave 0·2008 Hg

0·1661 „ 4·2 cc. N_2 at 31°C and 756 mm. pressure.

Found: Hg= 53·16 ; N=2·73; Calc. Hg=53·41 ; N=3·73.

The above nitrite when treated with ethyl iodide as in the previous cases, yielded a semi-solid mass; it was treated with methyl alcohol to remove tarry matter. The residue was shaken up with warm acetone when a portion of it dissolved and the rest was obtained in the shape of white crystals, with a faint yellow tint, (m.p. 107-108°) (=α). The filtrate was allowed to evaporate spontaneously after addition of a few drops of methyl alcohol, when successive crops were deposited, having the m. p. 118° (=β). The latter were unctous to the touch and resembled boracic acid. Both the compounds had the formula $(C_2H_4)_4S_3.HgI_2.2EtI$.

α—Variety:–

0·3755 gave 0·0783 Hg and 0·3650 AgI

0·1563 gave 0·0827 CO_2 and 0·0504 H_2O

β–variety:–

0·2826 gave 0·0590 Hg and 0·2704 AgI

Found:–

α–variety:– Hg=20·88; I=52·70; C=14·43; H=3·58.

β–variety:– Hg=20·86; I=52·23.

Calc. Hg=20·54; I=52·16; C=14·78; H=2·66.

Ethylene mercaptan and mercuric nitrita.

The dinitrite conformed to the formula $(C_2H_4S_2HgNO_2)_2$

0·1019 gave 0·0708 HgS

0·1960 gave 6·8 cc N_2 at 35° and 755 m.m.

Found:– Hg=59·88; N=3·64

Calc. Hg=59·17; N=4·14.

Ethylene mercaptide nitrite and ethyl iodide.

The product of interaction was a semi-solid mass; it was first treated with a small quantity of warm methyl alcohol, which dissolved out the colouring and tarry matter leaving a crystalline residue. The latter was refluxed with acetone for a few minutes. The insoluble portion consisted of beautiful yellow crystals, m.p. 151° sharp. The acetone filtrate on cooling gave the first crop consisting of a slight mixture of the above as also of the next compound (m.p. 121°) The second crop was also impure. The third crop had the m.p. 121° sharp. The mother liquor on further evaporation gave crystals having the m. p. 107; this product is exceedingly soluble in acetone and hence it is the last to crystallise out.

The compounds with the m.p.'s 151° and 107° respectively had indentical compositions and are in fact geometrical isomers. They both conform to the formula, $(C_2H_4)_2S_4.2HgI_2.2EtI$.

Analysis of the isomer, m.p. 151°

0·4468 gave 0·1294 Hg and 0·4544 AgI.

Found : Hg=28·96; I=55·46; Calc. Hg=28·49; I=54·28;

Analysis of the isomer, m.p. 107°

0·2364 gave 0·0661 Hg and 0·2401 AgI;

Found : Hg=27·96; I=54·88;

Analysis of the product, m. p. 121°, of the formula

$(C_2H_4)_2S_4.HgI_2.2EtI$.

0·3349 gave 0·0681 Hg and 0·3272 AgI;

Found: Hg=20·34; I=52·80; Calc. Hg=21·05; I=53·48;

Triethylene trisulphide and platinic chloride.

The components were added in alcoholic solution when a pale yellow crystalline precipitate was obtained; it was washed as usual dried and treated with carbon bisulphide to free it from any trace of sulphur which might be present. The crystalline product had the formula $(C_2H_4)_3S_3.PtCl_3$ (=A). The alcoholic mother-liquor on slow evaporation gave successive crops; the first and second of which were objected as being slightly contaminated with the above salt. The third and fourth crops were pure, and conformed to the formula $(C_2H_4)_3S_3.PtCl_4$ (=B)

Analysis of (A.)

Prep. I. 0·0660 gave 0·0256 Pt and 0·0604 AgCl and 0·0650 $BaSO_4$

Prep. II. 0·1063 gave 0·0460 Pt; 0·1020 AgCl and 0·1162 $BaSO_4$

Found: Pt=42·66,43.28; Cl=24·89,23·74; S=14·88, 15·01.

Calc. Pt=43·63; Cl=23·59; S=14·17.

Analysis of (B.)

Prep.I. 0·0330 gave 0·0123 Pt; 0·0354 AgCl and 0·0464 $BaSO_4$.

Prep.II. 0·0250 gave 0·0093 Pt; 0·2730 AgCl and 0·0305 $BaSO_4$

Found: Pt=37.27, 37·21; Cl=26·53, 27·01; S=19·31, 19·21.

Calc. Pt=37·96; Cl=27·06; S=18·49.

Chemical Laboratory,
College of Science,
University of Calcutta.

From the Transactions of the Chemical Society, 1921. Vol. 119.

CLXXXVI—*The Molecular Conductivity of some Sulphonium Compounds in Acetone.*

By Sir Praphulla Chandra Rây and Kalikumar Kumar.

In recent years one of the authors has prepared a large number of new sulphonium compounds which are insoluble in water but are generally soluble in acetone. The acetone solutions of these compounds are fairly good conductors, indicating that the substances behave as salts. An investigation of the conductivity of these compounds was therefore undertaken, with results which are tabulated below. The molecular conductivity increases regularly with dilution and its magnitude is similar to that observed for potassium iodide. These compounds appear, therefore, to be true salts having molecular weights represented by their respective chemical formulæ, and dissociating as uni-univalent electrolytes, possibly into a negative iodine-ion and a complex positive ion containing the sulphur chain.

The sulphonium compounds used in the present investigation have already been described (T., 1916, **1909**, 1, 135, 606; 1919, **115**, 551, 1152).

Merck's chemically pure acetone was used. It was further purified by rectification over anhydrous calcium chloride, and had the constant b. p. 55·8°; its specific conductivity was $3·3 \times 10^{-6}$ reciprocal ohms at 27°. The temperatures of the different experiments were between 26·8° and 28·8°, and the results are expressed for the temperature 27·8°.

The results obtained in the conductivity experiments are tabulated below, where V is the volume in litres containing a gram-molecular weight of the salt.

24414

Conductivity of Sulphonium Compounds in Acetone Solution at 27·8°.

Compound.	$V=64$.	$V=128$	$V=256$.	Remarks.
(1) $EtMeS_2,MeI,2HgI_3$	—	168·1	181·9	One hexad and one tetrad sulphur atom.
(2) $Et_2S_2,EtI,2HgI_2$ ...	114·5	135·4	144·3	
(3) Et_2S,MeI,HgI_2 ...	124·2	133·6	142·4	One hexad sulphur atom.
(4) Et_2S,EtI,HgI_2 ...	—	125·1	139·4	
(5) Et_2S,PrI,HgI_2 ..	113·3	121·7	136·9	
(6) Et_2S,BuI,HgI_2 ...	113·0*	120·5	132·3	*At $V=63·8$.
(7) Me_2S_2,MeI,HgI_2 ...	—	140·4	149·0	Chain compound with two tetrad sulphur atoms.
(8) Me_2S_2,EtI,HgI_2 ...	126·9	140·3	—	
(9) $MeEtS_2,EtI,HgI_2$ ...	127·0	139·0	—	
(10) Et_2S_2,EtI,HgI_2 ...	119·7	132·4	142·8	
(11) Pr_2S_2,MeI,HgI_2 ...	117·6	129·3	140·3	
(12) Et_2S_2,PrI,HgI_2 ...	112·2	125·3	133·6	
Potassium Iodide ...	—	115·5	130·5	At 25°.

Discussion.

The disulphonium chain compounds with one hexad and one tetrad sulpdur atom give the best conducting solutions in acetone. They are followed by the monosulphonium hexad compounds, whilst the disulphonium compounds with two tetrad sulphur atems give the smallest conductivities, comparable with the values for potassium iodide in acetone solution.

The conductivities of similar compounds diminish, without exception, with increasing molecular weight, for example, on comparing the compounds numbered (1) and (2) it will be found that the replacement of two methyl groups by two ethyl groups lowers the conductivity from 168·1 to 135·4. This relation between conductivity and molecular woight is borne out by the values for the compounds (3), (4) and (5), and these for the compounds (9), (10) and (11).

The compounds (11) and (12) have identical molecular weights, but show an appreciable difference in their conducting power. This is probably due to the relative difference in the magnitude of the aliphatic radicles of the two compounds. Thus the methyl and propyl radicles differ by two methylene groups, whereas ethyl and propyl differ by only one.

Chemical Laboratory,
University College of Science, Calcutta.

[*Received, June 28th, 1921*].

Varying Valency of Platinum with respect to Mercaptanic radicals.

By Sir Praphulla Chandra Rây.

The interaction of platinic chloride and ethyl mercaptan results in the formation of the chloro-mercaptide $(EtS)_2PtCl$ in which platinum was shown to function as trivalent (Trans. Chem. Soc., 1919, 115, 872). Recently diethyl disulphide was likewise treated with platinic chloride and the resulting product has been found to conform to the same formula. During the reaction chlorine is evolved. The mode of formation may be represented as follows :—

$$Et-S-S-Et + PtCl_4 \longrightarrow (EtS)_2PtCl.$$

The scission evidedtly takes place as shown by the dotted line and as platinum parts with some of the chlorine atoms the two etho-sulpho-groups attach themselves to it.

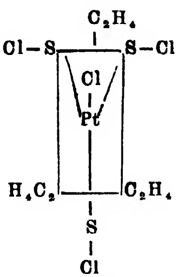
A more rational explanation seems to be, however, that on account of its greater affinity for sulphur the platinum not only parts with three of its chlorine atoms but has its additional latent valencies revived, the chloro-mercaptide being in reality a derivative of pentavalent platinum and a sulphonium compound as well, thus :—

$$\begin{array}{l} Et-S \\ \quad\ \ | > Pt-Cl \\ Et-S \end{array}$$

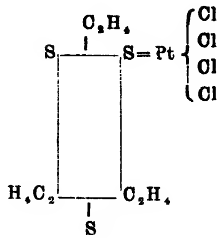
when the reaction is between ethyl mercaptan and platinic chloride, the liberated halogen acts upon the former generating diethyl disulphide. It, therefore, makes no difference whether ethyl mercaptan or the disulphide is used.

The penta as also the hexavalency of platinum is also established by the compounds having the empirical formula $(C_2H_4)_3S_3PtCl_4$ add

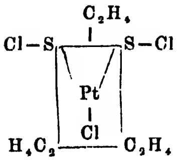
$(C_2H_4)_2S_3PtCl_4$ already described. These have evidently the formulæ



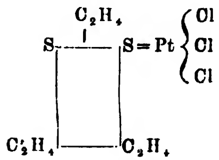
or



and



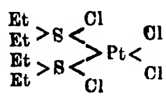
or



and are in reality sulphonium derivatives which the author has been investigating during the last seven years. It has recently been proved by the measurement of their conductivity in acetone solution that the so-called additive compounds *e.g.*, Et_2S_2,HgI_2,EtI are "atomic" in structure and their ionisation is of the same order as that of potassium iodide (T., 1921, **119**, 1643). Diethyl sulphide also yields with platinic chloride two compounds having the empirical formulæ $(Et_2S)_2PtCl_4$ and $(St_2S)\ PtCl_2$ both of them are chloromercaptides to the former may be assigned either of the two formulæ given below :

$$\begin{matrix}Et_2S\\Et_2S\end{matrix}\!>PtCl_4$$

or



according as platinum is regarded ar functioning as octa-or tetravalent. This sulphonium derivative has been obtained in the shape of well-defined crystals and has been proved in acetone solution to be a non-electrolyte. Cryoscopic molecular weight determination in benzene solution also confirms the same conclusion. The other compound which is also crystalline can only have one formula, namely ;

$$\begin{matrix}Et_2S\\Et_2S\end{matrix}\!>PtCl_2$$

the platinum functioning as hexavalent.

1 : 4 thiazan, $S<\begin{smallmatrix}C_2H_4\\C_2H_4\end{smallmatrix}>NH$ yields with hydrogen chloride the expected hydrochloride $C_4H_9NS.\ HCl$. Davies finds that the compound with platinic chloride has the formula $B.\ HCl.\ PtCl_4$ (Trans., C.S. 1920, *117*, 298) and he erroneously regards it as the platini-chloride of the base; had it been so it should have conformed to the formula $(B.\ HCl)_2\ PtCl_4$. The compound in question is evidently a chloro-mercaptide having the formula $Cl_4Pt{=}S<\begin{smallmatrix}C_2H_4\\C_2H_4\end{smallmatrix}>NH{\cdot}HCl$ in which platinum behaves as hexavalent.

Direct evidence of the variation in the valency of platinum has been obtained by the interaction of platinic chloride with the following mercaptans or rather their potassium salts which have been found to be more reactive;—(1) 2-thiol-5-thio-4-phenyl-4 : 5-dihydro-1 : 3 : 4-thiodiazole, $\begin{matrix}N(Ph)-N\\ | \qquad\quad \\ CS\text{———}S\end{matrix}\!\!>C{\cdot}SH$.

(2) Dithioethylene glycol, $\begin{matrix}CH_2SH\\ | \\ CH_2SH\end{matrix}$.

Trivalent Platinum.

The product of interaction in each case has been found to be a well-defined compound which admits of reproduction under the conditions of formation, which have been studied with great care. It has already been shown that if to a concentrated aqueous solution of the potassium salt of the above thiodiazole a dilute solution of chloroplatinic acid is added in a thin steam a product is obtained which consists of a mixture of equal proportions of trivalent and divalent platinum (Trans., Chem. Soc. 1919, *115*, 875). By adjusting the proper strengths of the perticipants the compound of trivalent platinum has now been isolated in a state of purity. By slightly modifying the process, the corresponding chloro-mercaptide has also been obtained.

Tri-, Tetra-, Penta-Hexa-, and Octavalent Platinum.

The mercaptan most suitable for the preparation of this series has been found to be dithioethylene glycol; although it is a dithiol it has only one atom of hydrogen of the—SH group replaceable by potassium. The substitution of an atom of potassium seems to exercise a sort of inhibitory influence on the second hydrogen atom

of the thiol group;—in fact, the latter becomes so inert as not to be vulnerable by the chlorine atoms of platinic chloride. Even chloropicrin, which is a very reactive agent yields with the mercaptide the compound

$$\left(C_2H_4<\begin{matrix}S\\SH\end{matrix}\right)_3\equiv C-O-C-\left(\begin{matrix}S\\HS\end{matrix}>C_2H_4\right)_3$$ (Trans., Chem. Soc 1919, *115*, 1308).

In the present instance the reaction takes place as follows :—

$$xC_2H_4<\begin{matrix}SH\\SK\end{matrix}+PtCl_4\longrightarrow Pt\left(C_2H_4<\begin{matrix}SH\\S\end{matrix}\right).$$

where x=3, 4, 5, 6 or 8. By using solutions of well-defined strengths of the reactants as also regulating the temperatures of the reactions, compounds have been obtained in which platinum functions as tri-,tetra-,penta-,hexa-and octa-valent respectively. For instance, by using solutions of platinic chloride as also of the mercaptide of definite strengths at the temperature of the laboratory (25-30°) haxavalent platinum compounds have been invariably obtained; again by properly varying the concentrations of the parent solutions at the some temperature, pentavalent compounds have been secured If, however, instead of changing the strength of the above solutions, the temperature of the solution of K-salt be reduced to 5-15° (the platinic chloride solution being kept between 25-30°) only octavalent compounds are produced; similarly, by regulating the range of temperature between 60-65° pentavalent compounds are formed ; at about 80° tetravalent ones are obtained, whereas at about 100° the product is uniformly trivalent At intermediate temperatures mixtures are obtained These reactions have been repeated in almost all cases for two to three dozen times under the above conditions with identical results In fact, it has been well-established that the particular valency which platinum will assume is a function of either of the two variables, concentration or temperature.

This might well have been expected from a consideration of the kinetic theory of molecules As increase of temperature has the effect of accelerating the mobility of the molecules the platinum atom is not placed in a favourable position in attracting to itself as many mercaptanic radicals as it is capable of Conversely, by lowering the temperature of the solution the molecular velocity diminishes and consequently the platinum atom is in a position to develop its

maximum group valency. It is a well-known fact that increase of temperature tends to dissociate the complex or heavier molecules into simpler ones, and the particular instance here in the case of platinum whose complexity diminished with increasing temperature falls within the line of this general observation. In other words the higher the temperature of reaction the lower the valency of platinum.

The place of platinum in the periodic table would naturally lead us to expect that like iridium it should behave as trivalent and like osmium as tetra-, hexa-, and octa-valent. Ruff and Tschirch have shown that the formation of the fluorides OsF_4, OsF_6, OsF_8 depends upon the temperature, the rate of flow of the fluorine current and the particular physical conditions of osmium (Ber., *46*, 929) and this fact is fairly well borne out in the case of the platinum derivatives which form the subject of the present communication.

It is not easy to give equations of the reactions involved in the formation of the above compounds and to account for the variation in the valency of platinum. The interaction of diethyl disulphide (Et_2S_2) and chloroplatinic acid results in the formation of the compound $(EtS)_2PtCl$ with evolution of chlorine. As diethyl disulphide itself results from the oxidation of ethyl mercaptan, the liberated chlorine cannot further act upon the former and hence the presence of any impurity in the shape of any oxidation product in this chloromercaptide is precluded. Platinum is evidently tervalent in this compound. When again chloroplatinic acid acts upon potassium-thiazole the platinum parts with all the chlorine atoms but contrary to expectation a derivative is obtained in which the metal functions as trivalent. Of course by doubling the formula the metal may be represented as tetravalent, when, however, it is borne in mind that the chlorides of iron and iridium, the analogues of platinum in the transitional group have the simpler formulae $FeCl_3$ and $IrCl_3$ respectively, it is scarcely likely that the platinum mercaptide should have a more complex formula. Similar arguments would go in support of the formula $\left(C_2H_4<^{SH}_{S}\right)_3Pt$ for the ethane dithiol derivative. The formation of quadrivalent platinum mercaptide may naturally be expected; it is not easy to account for the existence of penta—, hexa—and octavalent mercaptides. From considerations of the physical properties of chloroplatinic acid, *e. g.* ionisation, absorption spectrum and heat of

neutralisation, it is assigned the formula H_2PtCl_6. As an working hypothesis the formation of the above compounds may be explained according to the following equations, although objection may be raised that no direct proof of the evolution of H_2 is available :

$$5\left(C_2H_4<^{SH}_{SK}\right)+H_2PtCl_6=Pt\left(C_2H_4<^{SH}_{S}\right)_5+5KCl+HCl+H \quad \ldots\ldots\ldots(1)$$

$$6\left(C_2H_4<^{SH}_{SK}\right)+H_2PtCl_6=Pt\left(C_2H_4<^{SH}_{S}\right)_6+6KCl+H_2 \quad \ldots\ldots(2)$$

$$6\left(C_2H_4<^{SH}_{SK}\right)+2\left(C_2H_4<^{SH}_{S[H]}\right)+H_2PtCl_6$$

$$=\left(C_2H_4<^{SH}_{S}\right)_8Pt+6KCl+2H_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

In the last equation the two molecules of free mercaptan are derived from the hydrolysis of two molecules of potassium mercaptide, the motive of the reaction being the affinity of the sulphur. atom of the monovalent radical $C_2H_4<^{SH}_{S-}$ for platinum, and hence in the formation of the octavalent compound the hydrogen atom as shown under bracket is displaced so as to enable the sulphur atom to enter into direct relationship with the metal, and satisfy its maximum valency. It is wellknown that certain radicals, simple or compound, enable certain elements to develop their maximum latent valencies. Thus sulphur which in regard to chlorine is either di—or tetravalent behaves as hexavalent when combining with fluorine or iodine; platinum in respect to the radical $C_2H_4<^{SH}_{S-}$ has thus varying valencies depending upon the temperature and concentration of the latter.

On the otherhand it might be urged that chloroplatinic acid would act as if it had the formula $PtCl_4.2HCl$; there is then the possibility of chlorine being liberated in some instances at any rate, which would go to oxidise a portion of ethane dithiol into $(C_2H_4S_2)_x$; the latter being insoluble in ordinary solvents might be expected to contaminate some of the products. The presence of this oxidation product is, however, highly improbable. Let it be supposed that the reaction is only quantitative in the case of quadrivalent platinum and

that the other products are mere mixtures. These according to the results of the actual experiments should be represented as

$$Pt\left(C_2H_4<^{SH}_{S}\right)_4+x,\ Pt\left(C_2H_4<^{SH}_{S}\right)_4+2x,\ Pt\left(C_2H_4<^{SH}_{S}\right)_4+4x$$

where $x=C_2H_4S_2$ *

in other words, the amount of the impurity occurs in exact multiple proportions of the molecule $C_2H_4S_2$. In mixtures, however, one should expect the impurity to occur in indefinite proportions.

Another equally untenable position to which the above supposition would lead us is that the lower the temperature of the reaction the larger the amount of the oxidation product, the maximum oxidation taking place at 5—15° and the minimum at 100° *i.e.* the temperature yielding the trivalent platinum compound. The presence of any accidental foreign substance seems thus to be out of question. Indeed the octadic character of platinum far from being an anomaly is to be welcomed as justifying its place in group viii of the Periodic System.

EXPERIMENTAL.

Trivalent Platinum.

(a) *Thiodiazole and Platinic Chloride.*—0·5652 g. of the potassium salt was disolved in 10 c.c. water and to the solution was added under constant stirring 2 c.c. of chloro-platinic acid, (1 c.c. =0.0301 Pt.). A flocculent orange precipitate was obtained which was washed with water dried in a vacuum desiccator, powdered and treated successively with hot alcohol and benzene (T., 1919, *115*, 875) to extract the impurities. (Found: Pt=22·62; S=31·88. $C_{24}H_{15}N_6S_9Pt$ requires Pt=22·59; S=33.03). If, however, dilute solutions of the reactants be used a chloromercaptide is formed (see under pentad platinum).

(b) *Dithioethylene Glycol and Platinic Chloride.*—The *modus operandi* is the same as described under "hexad preparation," the temperature of the mercaptide solution being, in this case, about 100°. (Found: Pt=41·05; S=39·66; C=15·24; H=3·31. $C_6H_{15}S_6Pt$ requires Pt=41·13; S=40·51; C=15·13; H=3·15 per cent.).

* Analysis cannot discriminate between $C_2H_4<^{SH}_{S}$ And $C_2H_4S_2$ the difference being only one atom of hydrogen.

Tetravalent Platinum.

(a) *Dithioethylene Glycol and Platinic Chloride.*—The tetravalent compound of platinum by the interaction of dithioethylene glycol and platinic chloride has already been described (loc. cit. p. 876); but in the present instance the potassium salt of the mercaptan has been used. By using the same strength of it as in the cases of tri-, penta-, hexa- and octa-valent derivatives, but increasing the temperature to 80°, the desired compound has been obtained. Found: Pt=34·34; S=45·68; C=16·57; H=3·33. $C_8H_{20}S_8Pt$ requires Pt=34·39; S=45·14; C=16·93; H=3·54) per cent.

(b) *Triethylene Trisulphide and Platinic Chloride.*—The product of interaction is a crystalline compound conforming to the formula $(C_2H_4)_3S_3PtCl_4$ in which platinum behaves either as tetred or hexad (see above). 24414

Pentad Platinum.

(a) *Diethyl Disulphide and Platinic Chloride.*—An alcoholic solution of the components is refluxed on the water bath for three to four hours. The solution which to begin with remains perfectly clear gradually turns turbid and an orange precipitate puts in an appearance while chlorine is continuously evolved. The compound has the empirical formula $(C_2H_5S)_2PtCl$ but in reality it is pentad derivative of the metal. The same compound is also formed in the cold when a concentrated alcoholic solution of the parent substances is set aside for 24 hours or more. Found: Pt=54·51; Cl=9·95; S=18·11 $Pt(C_2H_5S)_2Cl$ requires Pt=55·02; Cl=10·09; S=18·19 per cent.).

It is in fact the same compound which has already been described as formed by the interaction of ethyl mercaptan and chloroplatinic acid. If instead of this simple mercaptan one with a complex radical, *e.g.*, thiodiazole, be used a corresponding chloromercaptide is obtained.

(b) *Thiodiazole and Platinic Chloride.*—To 4 c.c. of diluted chloroplatinic acid (=0·0400 Pt) were added drop by drop under constant stirring 7·5 c.c. of dilute solution of the potassium salt (=0·1058 g. K-salt). The precipitate was washed with water, dried in vacuum and treated successively with alcohol and

benzene. (Found: Pt=28·19; Cl=4·86; $C_{16}H_{10}S_6N_4PtCl$ requires Pt=28·86; Cl=5·20 per cent.)

(c) *Triethylene Trisulphide and Platinic Chloride.*—The trisulphide has also been shown to yield another compound with platinic chloride, which has the formula $(C_2H_4)_3S_2PtCl_3$, one atom of sulphur being detached from the trisulphide during reaction. The platinum here may be regarded as tri-, or penta-valent.

(d) *Dithioethylene Glycol and Platinic Chloride.*—To 0·4330 g. of the potassium salt dissolved in 8·4 c.c. of water at the ordinary temperature (25·30°) were added under vigorous shaking 23 c.c. of platinic chloride solution (1 c.c. =0·0067 g. Pt). The light brown precipitate was treated as described under hexad platinum. The same compound was obtained when the experiment was conducted under the same conditions as given below under hexad platinum, the only variation being that the temperature was kept between 60—65° (Found: Pt=29·78; S=48·01; C=18·46; H=4·54. $C_{10}H_{25}S_{10}Pt$ requires Pt=29·54; S=48·50; C=18·13; H=3·78 per cent.).

Hexad Platinum.

As the mercaptide $C_2H_4(SH)(SK)$* is the parent substance from which most of the derivatives of platinum have been obtained and as it has not been described before, its preparation is given here in detail. A large excess of a concentrated solution of alcoholic potash is added to ethylene mercaptan and stirred with a rod for a minute when the liquid mixture solidifies *en masse.* It is rapidly filtered with the aid of the suction pump and washed with alcohol. Care should be taken to limit the use of alcohol for the potassium salt is appreciably soluble in that menstruum. It keeps clear only for about half an hour and soon begins to turn turbid owing to aerial oxidation and the formation of the disulphide

$$C_2H_4<\begin{matrix}S\\|\\S\end{matrix}$$

(a) *Dithioethylene Glycol and Platinic Chloride.*—The hexad derivative is the one which is almost invariably formed when about 12 c.c. platinic chloride solution (1 c.c.=0·0260g. Pt) is

* Analysis of the potassium salt. (Found K=29·99; $C_2H_5S_2K$ requires K=29·55 per cent.).

added to 0·5500g. of potassium salt of dithio-ethylene glycol dissolved in 15 c.c. of water at temperatures between 25—30°.† The granular light-brown precipitate is vigorously shaken and a large volume of water is added to the mixture to preclude the possibility of the formation of potasium chloroplatinate (K_2PtCl_6). It is washed first with water then with alcohol and finally with ether and dried in a vacuum over sulphuric acid. Found: Pt=26·00; S=51·40; C=18·94; H=3·40; $C_{12}H_{30}S_{12}Pt$ requires Pt=26·09; S=50·85; C=19·07; H=3·97 per cent.).

(b) *Diethyl S·lphide and Platinic Chloride.*—A concentrated alcoholic solution of the above two components was set aside for two to three days. The crystalline products which were obtained were dissolved in boiling alcohol. The crop which was deposited on cooling conformed to the formula $(Et_2S)_2PtCl_4$ and had the m.p. 198°. On concentration of the mother liquor the product of the formula $(Et_2S)_2PtCl_2$ m.p. 77° was obtained.‡ (Found: Pt=44·01; Cl=16·05; $C_8H_{20}S_2PtCl_2$ requires Pt=43·97; Cl=15·85 per cent.).

Octad Platinum.

(a) *Dithioethylene Glycol and Platinic Chloride*—The conditions of the experiment were exactly the same as described under hexad platinum, the only difference being that the temperature was kept down to 5—15°. (Found: Pt=21·00; S=54·79; C=19·97; H=3·89; $C_{16}H_{40}S_{16}Pt$ requires Pt=20·94; S=54·41; C=20·36; H=4·25 per cent.).

The absence of chlorine was proved in all these compounds.

(b) *Diethyl Sulphide and Platinic Chloride.*—The preparation of the compound $(Et_2S)_2PtCl_4$ has been described under hexad. (Found: Pt=37·64; Cl=27·20; $C_8H_{20}S_2PtCl_4$ requires Pt=37·96; Cl=27·36 per cent.).

M. W. by cryoscopic method (in benzene)=540; $C_8H_{20}S_2PtCl_4$ requires M. W. =519.

† The platinum chloride solution is delivered from a burette in a thin stream under vigorous agitation and the operation is finished in less than five minutes.

‡ Both these compounds have already been described but the method of formation is quite different (Jahres, 1888, I, 1419).

Triethylene Tri- and Tetra-Sulphides

Part III

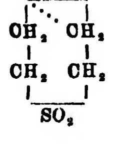
The Sulphones, Sulphinic and Sulphonic Acids of the Series.—Extension of Stuffer's Law

BY

Sir Prafulla Chandra RâY

The preparation and properties of these cyclic polysulphides and their derivatives have already been described (T., 1920, *117*, 1090; 1922, *121*, 1279). The compounds obtained by their oxidation and the hydrolytic decompositions undergone by these oxidation products form the subject of the present communication.

It has been shown that all sulphones, in which the sulphone groups are attached to two adjacent carbon atoms, can be saponified (Stuffer's Law, Ber., *26*, 1125). Baumann and Walter (1893, Ann., i, 458) have found that diethylene disulphide yields diethylene disulphone on oxidation with acid permanganate solution. This disulphone according to Stuffer's Law is hydrolysed into oxyethylsulphone ethylenesulphinic acid.



$$+ H_2O \longrightarrow HO{\cdot}CH_2{-}CH_2{-}SO_2{-}CH_2{-}CH_2{\cdot}SO_2H$$

The sulphinic acid contains a sulphone group between two carbon atoms and can further be hydrolysed in two different ways thus:

$$\text{I.}\quad \begin{array}{l} CH_2OH \\ | \\ CH_2\cdot SO_2\cdot CH_2\cdot CH_2\cdot SO_2H \end{array} + H_2O \longrightarrow \begin{array}{l} CH_2OH \\ | \\ CH_2OH \end{array} + \begin{array}{l} CH_2\cdot SO_2H \\ | \\ CH_2\cdot SO_2H \end{array}$$

$$\text{II.}\quad \begin{array}{l} CH_2OH \\ | \\ CH_2\cdot SO_2\cdot CH_2\cdot CH_2\cdot SO_2H \end{array} + H_2O \longrightarrow 2\left(\begin{array}{l} CH_2\cdot OH \\ | \\ CH_2\cdot SO_2H \end{array}\right)$$

The result of oxidation and subsequent hydrolysis on triethylene trisulphide is exactly analogous.

When triethylene trisulphide is oxidised with potassium permanganate in acid solution, triethylene trisulphone $(C_2H_4)_3S_3O_6$ is obtained, all the bivalent sulphur atoms in the ring becoming hexavalent.

$$\begin{array}{lcr} S & -C_2H_4- & S \\ | & & | \\ C_2H_4 & -S- & C_2H_4 \end{array} \xrightarrow{KMnO_4} \begin{array}{lcr} SO_2 & -C_2H_4- & SO_2 \\ | & & | \\ C_2H_4 & -SO_2- & C_2H_4 \end{array}$$

If, on the other hand, the oxidation is carried out with nitric acid, a compound with the formula $(C_2H_4)_3S_3O_4$, H_2O is produced.

$$\begin{array}{lcr} S & -C_2H_4- & S \\ | & & | \\ C_2H_4 & -S- & C_2H_4 \end{array} \xrightarrow{HNO_3} \begin{array}{lcr} S & -C_2H_4- & SO_2 \\ | & & | \\ C_2H_4 & -SO_2- & C_2H_4 \end{array}, H_2O$$

Only two of the three sulphur atoms become hexavalent while the third one remains unchanged (*vide infra*). It is not easy to account for this absence of reactivity in one of the atoms of sulphur when all the three are supposed to be symmetrically disposed with reference to the carbon atoms.

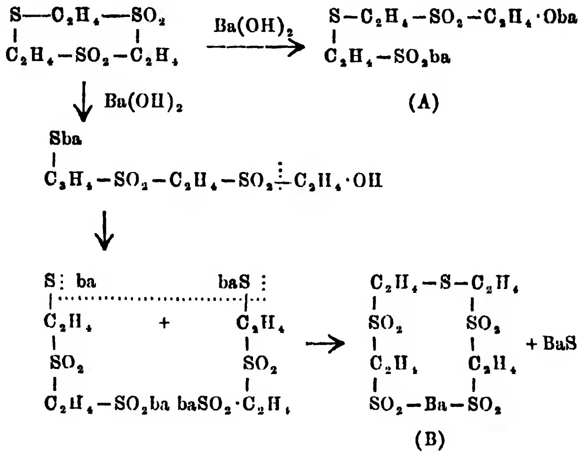
Triethylene trisulphone undergoes hydrolysis by means of baryta according to the following scheme:

$$\begin{array}{lcr} SO_2 & -C_2H_4- & SO_2 \\ | & & \vdots \\ C_2H_4 & -SO_2- & C_2H_4 \end{array} \xrightarrow{Ba(OH)_2} \begin{array}{l} SO_2-C_2H_4-SO_2ba \\ | \\ C_2H_4 \vdots -SO_2-C_2H_4\cdot OH \end{array}$$

$$\xrightarrow{Ba(OH)_2} \underset{(I)}{baSO_2\cdot C_2H_4\cdot SO_2\cdot C_2H_4\cdot OH} + \underset{(II)}{baSO_2-C_2H_4\cdot OH}$$

Compound (I) has been isolated in a pure condition but compound (II) could not be obtained in a sufficiently pure state for purposes of

analysis. The disulphone sulphide on similar treatment with baryta yields two products, as shown below:



It will be observed that barium sulphide is produced in course of this reaction. In fact, a continuous evolution of hydrogen sulphide takes place during the removal of excess of baryta by means of carbon dioxide. This can never be the case if all the sulphur atoms of triethylene trisulphide had been converted into sulphone. Hence, one of the sulphur atoms must have remained unacted during oxidation with nitric acid.[1] Again, it is very difficult to offer any satisfactory explanation for the dibasic character of the compound (A). It might be assumed that the proximity of the sulphone group renders the hydroxyl group acidic thus enabling the hydrogen atom of the hydroxyl to be replaced by barium. Even if that is so it is not clear why in the compound, oxyethylsulphone ethylenesulphinic acid, a decomposition product of triethylene trisulphone (*vide supra*), a similar phenomenon is not observed.

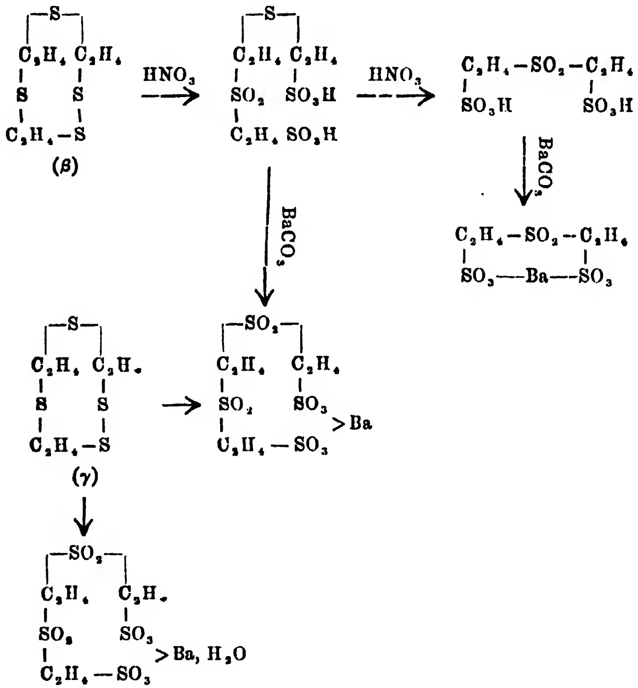
The β- and γ-triethylene tetrasulphides (*loc. cit.*) have also been similarly oxidised with nitric acid. Each of the sulphur atoms attached to two adjacent carbon atoms is oxidised to sulphone, whereas the link between two contiguous sulphur atoms is broken with the formation of a disulphonic acid. It may be mentioned here with advantage that another cyclic compound ethylene disulphide, the

[1] The oxidation products of ethylene sulphide described by Crafts' (Ann., 125, 123) are evidently different from those described in this paper.

product of oxidation of dithioglycol behaves exactly similarly on oxidation with nitric acid. Scission takes place at the link between the two adjacent sulphur atoms, resulting in the formation of ethylene disulphonic acid, thus:

$$C_2H_4<\begin{matrix}S-S\\ S-S\end{matrix}>C_2H_4 \longrightarrow 2\,C_2H_4<\begin{matrix}SO_3H\\ SO_3H\end{matrix}$$

The oxidation of β- and γ-tetrasulphides takes place according to the following manner:



The above reactions will show that the same barium triethylene disulphone disulphonate is obtained by the oxidation and subsequent neutralisation with barium carbonate of both the β- and γ-varieties. In the case of the β-compound, the free acid further breaks down into diethylene sulphone disulphonic acid.

By the oxidation of β- and γ-triethylene tetrasulphides by means of potassium permanganate a tetrasulphone is obtained which form a series of double salts with metallic sulphates. These will form the subject of a further communication.

EXPERIMENTAL

Oxidation of Triethylene Trisulphide with Potassium Permanganate.

The trisulphide was suspended in small quantity of water and treated with a concentrated solution of potassium permanganate and a few c.c.'s of dilute sulphuric acid. A vigourous reaction ensued with evolution of heat. The mass was cooled and the process repeated till there remained an excess of permanganate solution. The product was then finally heated on the water bath for nearly 3-4 hours. After completion of the oxidation, the manganese peroxide was brought into solution by passing a current of sulphur dioxide through the liquid. The sulphone which remained insoluble was filtered and crystallised from concentrated nitric acid. It is insoluble in all the ordinary organic solvents and sublimes at a high temperature. (Found: $S=34 \cdot 25$; $C=24 \cdot 93$;[1] $H=4 \cdot 20 \cdot (C_2H_4)_3S_3O_6$ requires $S=34 \cdot 78$ $C=26 \cdot 08$; $H=4 \cdot 35$ per cent.).

The trisulphone was treated with an excess of baryta water and heated over a naked flame. It dissolved in the course of nearly half an hour. The solution was then cooled and the excess of baryta removed by means of carbon dioxide. The filtrate was concentrated to a small bulk and treated with an equal volume of alcohol. A syrupy liquid was obtained which on keeping solidified into crystalline mass. The solid was filtered off washed with alcohol and dried in a vacuum desiccator. (Found: $Ba=25 \cdot 42$; $S=23 \cdot 66 \cdot C_4H_9O_5S_2ba$ requires $Ba=25 \cdot 42$; $S=23 \cdot 75$ per cent.).

The mother liquor from the above salt was concentrated and a large volume of alcohol added to it. A thick syrup was obtained, which solidified on keeping in a vacuum desiccator. This salt is extremely hygroscopic and could not be sufficiently purified for analytical purposes.

Oxidation of Triethylene Trisulphide with Nitric Acid.

The trisulphide was treated with an excess of fuming nitric acid in a sealed tube and heated to 100° for about an hour. The tube was then opened and the nitric acid evaporated to dryness. The residue was dissolved in hot water from which the triethylene disulphone sulphide

[1] The substance sublimes at a high temperature. It is generally found that in such cases a trace of it escapes oxidation; hence the percentage of carbon comes out a little too low.

crystallised out in colourless well-defined prisms.[1] It chars at about 250°. (Found: C=27·33; H=4·76; S=37·09·$(C_2H_4)_3S_3O_4$, H_2O requires C=27·48; H=5·34; S=36·64 per cent.).

The disulphone sulphide was mixed with an excess of baryta water and heated over a naked flame for about 15-20 minutes. Quantities of barium sulphide, sulphite and carbonate were formed during the reaction. The excess of baryta was then removed by passing carbon dioxide through the liquid. The filtrate was concentrated to a small bulk and an equal volume of alcohol added to it. A white crystalline salt separated out. This was filtered off and dried in a vacuum. (Found: Ba=34·16; S=23·77·$C_6H_{12}O_5S_3Ba$ requires Ba=34·42; S=24·12 per cent.). The mother liquor was further concentrated to a thick syrup, which crystallised on keeping in a vacuum desiccator for a few days. (Found: Ba=23·89; S=26·46·$C_8H_{16}O_8S_5Ba$, $3H_2O$ requires Ba=23·18; S=27·07 per cent.).

Oxidation of β-Triethylene Tetrasulphide with Nitric Acid.

The tetrasulphide was heated with fuming nitric acid in a sealed tube at 100° for about an hour. After opening the tube the nitric acid was completely evaporated off. The residue was refluxed with acetone in which a part of it dissolved. The insoluble portion was dissolved in water, neutralised with barium carbonate and filtered. On concentrating the filtrate a crystalline barium salt was obtained. This was dried in a desiccator. (Found: Ba=26·48; S=25·46·$C_6H_{12}O_{10}S_4Ba$ requires Ba=26·92; S=25·15 per cent.).

The acetone solution was evaporated to dryness and the viscous liquid thus left dissolved in a small quantity of water. The aqueous solution was neutralised with barium carbonate and filtered. The filtrate on evaporation left behind well-defined white crystals. (Found: Ba=32·89; S=22·49·$C_4H_8O_8S_3Ba$ requires Ba=32·85; S=23·02 per cent.).

Oxidation of γ-Triethylene Tetrasulphide with Nitric Acid.

The oxidation was carried out in exactly the same way as in the case of the β-modification. The residue left after driving away the last traces of nitric acid was dissolved in water and neutralised with

[1] The experiment was repeated several times with identical results. In one instance, however, a small quantity of an insoluble sulphone (as is obtained in the case of oxidation with permanganate) was produced.

excess of barium carbonate. Two salts were formed. One remained in solution (A) while the other was precipitated along with the excess of barium carbonate. The insoluble mass was therefore extracted with water, which on cooling gave a crop of barium salt. (Found: Ba=25·49; S=24·04·$C_6H_{12}O_{10}S_4Ba$, H_2O requires Ba=25·90; S=24·10 per cent.).

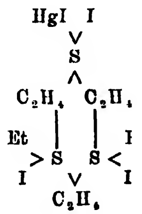
The solution (A) was concentrated and on being cooled gave crystals of a barium salt. (Found: Ba=26·35; S=25·20· $C_6H_{12}O_{10}S_4Ba$ requires Ba=26·92; S=25·15 per cent.).

Oxidation of Ethylene Disulphide with Nitric Acid.

The oxidation was carried in a sealed tube as before. After removal of the last traces of nitric acid, the syrupy liquid left behind was dissolved in water and neutralised with barium carbonate. The barium salt was precipitated from the concentrated solution by means of alcohol. (Found: Ba=42·26; $C_2H_4O_6S_2Ba$ requires Ba=42·16 per cent.).[1]

CHEMICAL LABORATORY,
UNIVERSITY COLLEGE OF SCIENCE, CALCUTTA.

[1] After the above investigation had been ready for communication my attention was drawn to a paper by Bennet (Trans. Chem. Soc., 1922, *121*, 2144, *foot-note*). Cryoscopic determination of the molecular weight of the cyclic sulphide in benzene as also the determination of vapour density by Hofmann's method no doubt justifies the formula $(C_2H_4S)_2$; there are however difficulties in the way of accepting this formula judged from the point of view of its simple break up according to Stuffer's Law. Moreover, the formation of the compounds $(C_2H_4)_3S_3PtCl_4$ and $(C_2H_4)_3S_2PtCl_3$ (T., 1922, *121*, 1283) also justifies the formula $(C_2H_4)_3S_3$. The dithian formula $(C_2H_4)_2S_2$ cannot offer any rational explanation of the formation of the compound $(C_2H_4)_3S_3 \cdot HgI_2 \cdot 2EtI$, (l. c. p. 1281). Its constitution can be well-represented as

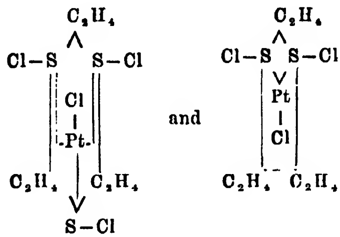


In this case each divalent sulphur atom becomes quadrivalent yielding a sulphonium derivative. While according to the dithian formula the compound would be

represented as

$$\left[\begin{array}{c}Et\\ I\end{array}\!>\!S\!<\!\begin{array}{c}C_2H_4\\ C_2H_4\end{array}\!>\!S\!<\!\begin{array}{c}HgI\\ I\end{array}\right]EtI \quad \text{or} \quad \left[\begin{array}{c}Et\\ I\end{array}\!>\!S\!<\!\begin{array}{c}C_2H_4\\ C_2H_4\end{array}\!>\!S\!<\!\begin{array}{c}Et\\ I\end{array}\right]\cdot HgI_2$$

which does not seem to be at all likely. The compounds $(C_2H_4)_3S_3PtCl_4$ and $(C_2H_4)_2S_2PtCl_3$ may also be structurally represented as



the platinum functioning as ter- and quadrivalent respectively the compounds at the same time being regarded as sulphonium derivatives. A critical examination of this point the author hopes to take up at an early date.

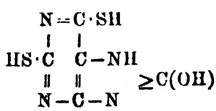
Some Mercaptans of the Purine Group

Part I

BY

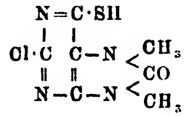
Sir Prafulla Chandra Rây, Gopâl Chandra Chakravarti and Prafulla Kumâr Bose

In the course of an investigation on trithiopurine, it was necessary to prepare 2:6-dichloro-8-oxypurine in quantity. It occurred to the present authors to study the action of potassium hydrosulphide on the latter compound with a view to the synthesis and study of the corresponding possible mercaptans. It was, however, found that a boiling alcoholic solution of potassium hydrosulphide did not react with dichloropurine in the way expected—the hydrosulphide being simply decomposed with the separation of the potassium salt of the oxypurine. An aqueous solution of potassium hydrosulphide at 100° reacts with the oxypurine yielding a mixture of various products from which the mercaptan could not be isolated in a pure condition. At a higher temperature, however, and under pressure, both the chlorine atoms were replaced by 'thiol' groups yielding 2:6-dithio-8-oxypurine.



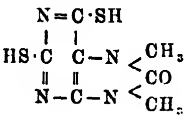
The action of potassium hydrosulphide on several chloropurines has been studied by Fischer and his co-workers (Ber., *31*, 431). They have shown that in the cases of trichloropurine, 7-methyl-2:6-dichloropurine and others the reaction with potassium hydrosulphide proceeds very smoothly even in the cold. The difficulty observed in our case in replacing the chlorine atoms by 'thiol' groups was supposed to be due to the presence of the acidic hydroxyl grouping in the position 8. Accordingly, 2:6-dichloro-8-oxypurine was methylated to 7:9-dimethyl-2:6-dichloro-8-oxypurine (Fischer, Ber., *30*, 2208),

whereby the substance completely lost its acidic nature. On treatment with hot alcoholic potassium hydrosulphide, one of chlorine atoms of the dimethyl derivative was replaced by a 'thiol' group, thus showing that the chlorine atom is rendered more labile by the suppression of the acid character of the oxypurine. The thiopurine has probably the constitution,

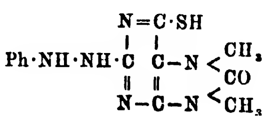


as Fischer (Ber., *30,* 2226 ; *31,* 104 ; *31,* 431, etc.) has shown that of the two chlorine stoms in positions 2 and 6, the one at 6 is more reactive than that at 2. The extreme reactivity of substituents in position 6 is further corroborated by the fact that the methyl ether of the above-mentioned thiopurine, when boiled with potassium hydrosulphide in alcohol, is hydrolysed to the parent mercaptan instead of yielding the 2-thiol derivative as expected.

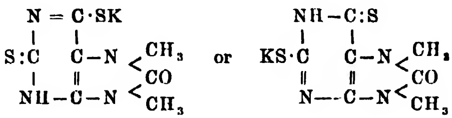
2 : 6-Dithio-7 : 9-dimethyl-8-oxypurine



has been obtained from both 2:6-dichloro-7:9-dimethyl-8-oxypurine and 2-chloro-6-thio-7:9-dimethyl-8-oxypurine by heating with aqueous potassium hydrosulphide under pressure at 130°. These mercaptans possess strongly acid character and form beautifully crystalline potassium and sodium salts. The hydrogen atoms of the 'thiol' groups are readily replaced by alkyl groups, producing alkyl ethers. 2-chloro-6-thio-7:9-dimethyl-8-oxypurine gives with mercuric chloride a chloromercaptide which crystallises with four molecules of ethyl alcohol; the chloromercaptide further combines with one molecule of mercuric chloride, as might well have been expected from the basic property due to the presence of several tertiary nitrogen atoms. With phenyl-hydrazine the chlorine atom is replaced, with the formation of the compound.



It is known that tautomerism of the type $\cdot C(OH)=N \rightleftharpoons CO-NH$ is exhibited by many oxypurines, but a similar tautomeric behaviour in the case of thiopurines has not been noticed by any investigator. The dithiopurines described above are expected to give the corresponding dipotassium salts but they actually produce the monopotassium ones. It seems, therefore, very likely that one of the 'thiol' groups is in the keto-form, the other being reactive as usual. Thus,



The fact that titration with iodine indicates the presence of two 'thiol' groups in the molecule, might militate against the existence of the thiopurines in the ketonic form, but instances of the enolising action of halogens are not rare. (*Cf.* Dawson and Powis, T., *95*, 1860; *97*, 2048; *101*, 1503. Lapworth, *ibid.*, *85*, 30 etc.). It is thus evident that keto-enol tautomerism of the type indicated above may exist in the thiopurines, though it differs from the corresponding oxypurines in degree but not in kind.

EXPERIMENTAL

2 : 6-Dithio-8-oxypurine.—One molecule of 2 : 6-dichloro-8-oxypurine was just dissolved in a dilute solution of caustic potash. To it was added 6 to 7 molecules of a 10% solution of aqueous caustic potass saturated with sulphuretted hydrogen in the cold. The tube was opened when cool and the solution filtered and acidified with a slight excess of dilute hydrochloric acid. The precipitate was collected, washed with water and dissolved in the least quantity of a hot saturated solution of sodium carbonate. The solution was decolorised by animal charcoal and filtered hot. On cooling the sodium salt crystallised out in fine colourless needles. (Found: Na = 9·77. $C_5H_3ON_4S_2$ requires Na = 10·36 per cent.).

On dissolving the sodium salt in water and acidifying with dilute hydrochloric acid, the free mercaptan is precipitated in very small yellow nodules. (Found: S = 31·19; N = 27·78; SH (by titration with iodine) = 32·30·$C_5H_4ON_4S_2$ requires S = 32·00; N = 28·00; SH = 33·00 per cent.).

The mother liquor from the above sodium salt, on being treated with dilute hydrochloric acid gave a small quantity of a precipitate, which was filtered off. The filtrate, when acidified with concentrated hydrochloric acid gave a crop of not very well-defined crystals (needle-shaped), insoluble in ordinary solvents but slightly soluble in boiling water from which it separates out as a flocculent precipitate. Attempts were made to purify it by dissolving it in sodium carbonate solution and acidifying with a large excess of concentrated hydrochloric acid. A crystalline product was obtained. (Found: N=26·62; Cl=19·70, 17·79; S=29·99, 31·14 per cent.). The percentage of nitrogen shows that the ring evidently remains intact—but the high percentage of chlorine indicates that it is a mixture of varying proportions of 2:6-dithio-8-oxypurine and 2-chloro-6-thio-8-oxypurine.

Dimethyl ether of 2:6-Dithio-8-oxypurine.—was prepared by heating on the waterbath under reflux a solution of the sodium salt of the mercaptan in water with an excess of methyl iodide. The precipitate was collected and washed with water. From acetone solution it separates in the form of minute crystals melting with decomposition at 285-88°. (Found: N=25·08; $C_7H_8ON_4S_2$ requires N=24·56 per cent.).

7:9-Dimethyl-8-oxy-2-chloro-6-thiopurine.—5 grams of potassium hydroxide in 100 c.c. of alcohol was saturated with sulphuretted hydrogen in the cold. To it was added 10 grs. of finely divided 7:9-dimethyl-8-oxy-2:6-dichloropurine suspended in about 100 c.c. of alcohol. The mixture was boiled under reflux for nearly 15 minutes when a white crystalline precipitate consisting of a mixture of potassium chloride and the potassium salt of the mercaptan separated out. The precipitate was filtered, dissolved in water and again filtered from any insoluble impurities such as sulphides and treated with dilute hydrochloric acid in excess. The precipitated mercaptan was washed with hot water, dried and then twice refluxed with carbon disulphide to dissolve out any adhering traces of sulphur. It was finally purified by crystallisation from glacial acetic acid in which it is only slightly soluble. It is also very sparingly soluble in ethyl and amyl alcohol from which it separates out in the form of brownish-yellow rhombic crystals. (Found: S=13·85; Cl=16·36 and N=24·25·$C_7H_7ON_4SCl$ requires S=13·88; Cl=15·40 and N=24·29 per cent.). It is a strong monobasic acid and readily dissolves in dilute aqueous solutions of alkalies, alkali carbonates, and bicarbonates. It is perfectly

odourless, bitter to the taste and does not give the murexide reaction. When heated it decomposes without melting.

Potassium Salt.—The mercaptan was dissolved in a dilute boiling solution of alcoholic potash and filtered hot. On cooling colourless needles of the potassium salt separates out. (Found: K = 14·53. $C_7H_6ON_4SClK$ requires K = 14·53 per cent.).

Ammonium Salt.—The mercaptan was dissolved in a boiling solution of alcoholic ammonia and filtered. The filtrate was diluted with ether when fine silky crystals separated out.

Interaction of 7 : 9-Dimethyl-8-oxy-2-chloro-6-thiopurine and Mercuric Chloride.—The mercaptan was dissolved in hot amyl alcohol and filtered. To the hot solution was added with constant stirring an excess of an alcoholic solution of mercuric chloride. A white curdy precipitate was at once formed and filtered off when cool. The residue was extracted with hot amyl alcohol. The chloromercaptide remained behind as a white precipitate (A). Whereas the other product of reaction (*vide supra.*) crystallised out from the hot filtrate in colourless needles (B). This was recrystallised from amyl alcohol, washed with absolute alcohol and dried in a vacuum desiccator. The above experiment was repeated using an ethyl alcoholic solution of the mercaptan. The precipitate (C) was collected, washed with absolute alcohol and dried.

(Found: (A) C = 25·70;[1] H = 3·06; Hg = 30·40 and Cl = 10·47.
(B) Hg = 53·00. $C_7H_6ON_4SCl_4Hg_2$ requires Hg = 54·34 per cent.).
(C) Hg = 30·83. $C_7H_6ON_4Cl_2SHg$. $4C_2H_5OH$ requires C = 27·73; H = 4·62; Hg = 30·80 and Cl = 10·90 per cent.

It is thus evident that compounds (A) and (C) are identical and are pure chloromercaptides. The compound (B) is not formed when ethyl alcohol is used instead of amyl alcohol.

Disulphide.—The thiopurine was dissolved in aqueous potash and treated with a solution of iodine in potassium iodide till the brown colour of iodine persisted. The precipitate was collected and dried.

[1] The percentages of carbon and hydrogen come out slightly low, due to traces of alcohol escaping oxidation as is often found to be the case when compounds containing alcohol of crystallisation are analysed.

It is obtained as colourless needles from hot toluene melting at 259°. (Found : S=13·77 and Cl=15·92. $C_{14}H_{12}O_2N_8Cl_2S_2$ requires S=13·96 and Cl=15·46 per cent).

Methyl Ether of 7 : 9-Dimethyl-8-oxy-2-chloro-6-thiopurine.—The potassium salt of the mercaptan was dissolved in water and heated under reflux on the waterbath with excess of methyl iodide for 15 to 20 minutes. The ether separates out from the reaction mixture in needles. It was crystallised from absolute alcohol in which it is very soluble. *m.p.* 179°. (Found: N=20·49 ; $C_8H_9ON_4ClS$ requires N=22·90 per cent.).

The corresponding *Ethyl Ether* crystallises from 60% methyl alcohol in colourless needles melting at 133°. The *n-Propyl Ether* is soluble in hot absolute alcohol and melts at 120°. (Found: S=12·09. $C_{10}H_{13}ON_4ClS$ requires S=11·74 per cent.).

Interaction of 7 : 9-Dimethyl-8-oxy-2-chloro-6-thiopurine and Phenyl Hydrazine.—About 2 grs. of phenyl hydrazine hydrochloride and an equal quantity of sodium acetate was dissolved in about 25 c.c. of water. This was heated on the waterbath with about 0·3 gr. of the finely divided mercaptan for 7 to 8 hours with frequent stirring. The condensation product separated from the hot reaction mixture in the form of needles. It was filtered washed successively with hot water, alcohol and chloroform, and dried in a steam oven. It was found to be insoluble in all the ordinary organic solvents. (Found:N=28·16. $C_{13}H_{14}ON_6S$ requires N=29·37 per cent.).

7 : 9-Dimethyl-8-oxy-2 : 6-dithiopurine.—About 50 c.c. of an aqueous solution of potassium hydroxide (2N approx.) was saturated with sulphuretted hydrogen in the cold. 1 gr. of 7:9-dimethyl-8-oxy-2:6-dichloropurine was suspended in this solution and the mixture was heated in a sealed tube at 130-40° for three hours. The reaction product, a liquid, was filtered from any impurities and acidified with dilute hydrochloric acid, when a yellow precipitate was obtained. It was crystallised from glacial acetic acid. (Found: S=27·98 $C_7H_8ON_4S_2$ requires S=28·07 per cent.).

It is a yellow crystalline substance, melting with decomposition above 300°. It is a stronger acid than the monothiopurine described above. Like the monothiopurine it does not give the murexide reaction. This dithiopurine has also been obtained from 7 : 9-dimethyl-8-oxy-2-chloro-6-thiopurine in exactly the same way.

The Potassium Salt.—is best prepared by adding a cool and concentrated solution of aqueous potassium hydrosulphide to the finely divided mercaptan with vigorous shaking. The silky needles of the potassium salt are washed with absolute alcohol and dried in a vacuum desiccator. It reacts quantitatively with iodine. (Found: K=14·76 and SH=24·39. $C_7H_7ON_4S_2K$ requires K=14·66 and SH=24·81 per cent.).

The *dimethyl ether* crystallises from carbon disulphide in colourless prisms melting at 172-73°. (Found: N=21·28. $C_9H_{11}ON_4S_2$ requires N=21·87 per cent.). The *diethyl ether* separates from dilute alcohol in colourless needles melting at 104°. (Found: N=19·99. $C_{11}H_{16}ON_4S_2$ requires N=19·72 per cent.). The *dibenzyl ether* crystallises from a mixture of benzene and ether in rhombic prisms melting at 158°. (Found: N=14·06. $C_{21}H_{20}ON_4S_2$ requires N=13·72 per cent.). These ethers have been prepared in the manner employed in the case of the other purine ethers described above.

The Oxidation of Triethylene Tetrasulphide with Potassium Permanganate

BY

Sir P. C. Rây

The oxidation of triethylene tetrasulphide with nitric acid has been shown to result in the break up of the molecule with the formation of the corresponding disulphonic acid. Each of the sulphur atoms situated between a pair of carbon atoms is converted into a sulphone while fission takes place between the two contiguous sulphur atoms with the formation of the sulphonic acid. It seemed desirable to study the action of a less drastic oxidising agent like potassium permanganate, which might be expected to yield the corresponding tetrasulphone. This expectation has been realised. The sulphone, however, conbines with the manganous sulphate formed during the reaction and a stable compound of the formula $1\frac{1}{2}[(C_2H_4)_3S_4O_8]$, $MnSO_4$, $6H_2O$ is invariably formed. It has been found to be very sparingly soluble in the cold but dissolves readily in boiling water. By adding barium chloride to this boiling solution the corresponding compound with barium sulphate is at once thrown down. By a similar treatment compounds with the sulphates of strontium and calcium as also of potassium and lead have been obtained. With silver nitrate a compound with silver sulphate is formed. Corresponding double compounds with the sulphates of nickel, cobalt and copper have also been obtained by the addition of the respective chlorides. Peculiar interest attaches to some of these compounds, notably the combination of the sulphone with barium sulphate; on account of its extreme insolubility all attempts to combine this substance with other compounds have hitherto been unsuccessful. But by following this indirect* method, however, an additive compound with barium sulphate has been secured. The sodium compound could not be isolated in a pure condition by a method analogous to that for the preparation of the potassium compound: a special procedure was therefore adopted.

In all these compounds the proportion of the respective metallic sulphate to the sulphone holds simple integral relationship. In the case of the barium compound alone the components are in the simple ratio of 1 : 1 ; whereas they are in the ratio of 2 : 3 in the manganese, copper, cobalt and nickel compounds. The ratio is as 4 : 5 in the potassium, calcium, strontium, silver and lead double compounds. The sodium compound alone gives a ratio of 4 : 1 between sodium sulphate and the sulphone.

EXPERIMENTAL

The Oxidation of Triethylene Tetrasulphide with Acid Potassium Permanganate.—The tetrasulphide was treated with a mixture of a concentrated solution of potassium permanganate and dilute sulphuric acid in small quantities at a time and heated on the water bath. This process was continued till there remained a distinct excess of the permanganate. The oxidation was complete in the course of about four to six hours. The excess of permanganate was removed by passing a current of sulphur dioxide through the mixture. The solution became hot, leaving a residue of the unacted tetrasulphide, the latter was filtered off and the filtrate concentrated on the water-bath to nearly half its volume. On cooling, a colourless compound crystallised out. This was filtered, washed with cold water and dried. (Found: Mn=7·38 ; S=28·88. $1\frac{1}{2}[(C_2H_4)_3S_4O_8]$, $MnSO_4$, $6H_2O$ requires Mn=7·15; S=29·14 per cent.).

Calcium Compound.—The manganese compound was dissolved in boiling water and a concentrated solution of calcium chloride added to it. A white crystalline calcium compound was at once precipitated, which was filtered and dried. (Found: Ca=7·09; C=16·81. $1\frac{1}{4}[(C_2H_4)_3S_4O_8]$, $CaSO_4$ requires Ca=7·13 ; C=16·04 per cent.).

Strontium Compound.—This compound was prepared by a process similar to that used for the preparation of the barium compound. (Found: Sr=14·68 ; C=14·66. $1\frac{1}{4}[(C_2H_4)_3S_4O_8]$, $SrSO_4$ requires Sr=14·31 ; C=14·80 per cent.).

Barium Compound.—This compound was also prepared like the calcium compound. (Found: Ba=23·24 ; S=26·25. $[(C_2H_4$ $S_4O_8]$, $BaSO_4\cdot H_2O$ requires Ba=23·18 ; S=27·00 per cent.).

Lead Compound.—Lead chloride was dissolved in hot water and the solution added to a solution of the manganese compound in

boiling water. The precipitate was filtered, washed with hot water and dried. (Found: Pb=28·35. $1\frac{1}{4}[(C_2H_4)_3S_4O_8]$. $PbSO_4$ requires Pb=28·44 per cent.).

Silver Compound.—On adding a very concentrated solution of silver nitrate to a boiling solution of the manganese compound a white precipitate was at once thrown down. (Found: Ag=29.68. $1\frac{1}{4}[(C_2H_4)_3S_4O_8]$. $AgSO_4$ requires Ag=29·31 per cent.).

Copper, Cobalt and Nickel Compounds.—These products were obtained by adding a concentrated solution of the respective chlorides to a concentrated solution of the manganese double compound in hot water. The mixture on concentration and cooling gave crystals which were filtered, washed with a little cold water and dried. These compounds are fairly soluble in hot water. The copper compound is slightly bluish in colour. (Found: Cu=9·06; S=29·55. $1\frac{1}{2}[(C_2H_4)_3S_4O_8]$. $CuSO_4$, $4H_2O$ requires Cu=8·58; S=30·25. per cent.). The nikel compound is slightly greenish in colour. (Found: Ni=8·55; S=30·03. $1\frac{1}{2}[(C_2H_4)_3S_4O_8]$, $NiSO_4$, $4H_2O$ requires Ni=7·88; S=30·43 per cent.). The cobalt compound has a slight pink tint. (Found: Co=8·23; S=29·28. $1\frac{1}{2}[(C_2H_4)_3S_4O_8]$, $CoSO_4·4H_2O$ requires Co=8·00; S=30·39 per cent.).

Potassium Compound.—This compound was obtained as before by adding a concentrated solution of potassium chloride. There was, no precipitate while the solution was hot, but on cooling a crystalline compound separated out. Found: K=28·57. $1\frac{1}{4}[(C_2H_4)_3S_4O_8]$. K_2SO_4 requires K=29·07 per cent.).

The Sodium Compound.—An aqueous solution of sodium carbonate was gradually added to a solution of the manganese compound till manganous carbonate was completely precipitated. This was filtered and the filtrate concentrated to a small bulk. On adding alcohol to this concentrated solution a syrup was obtained, which crystallised on keeping in a vacuum desiccator. The crystals were dissolved in water and reprecipitated with alcohol. (Found. Na=18·26; S=25·87. $\frac{1}{4}[(C_2H_4)_3S_4O_8]$, Na_2SO_4,H_2O requires Na=18·77; S=26·12 per cent.).

PHYSICS

The Scattering of Light by Liquid Droplets, and the Theory of Coronas, Glories and Iridescent Clouds

BY

BIDHUBHUSAN RAY, M.Sc.,

Lecturer in Optics, Calcutta University.

1. Introduction.
2. Coronas and iridescent clouds.
3. Glories or brocken-bows and their explanation.
4. Intensity and polarisation of the scattered light.
5. Rigorous electromagnetic theory of diffraction by spherical particles.
6. Summary and conclusion.

1. *Introduction.*

A variety of interesting optical effects are known which arise from the scattering of sunlight by droplets of water suspended in the atmosphere and have been discussed by writers on meteorological science. A good account of these phenomena and a discussion of them, on the elementary principles of geometrical optics and wave theory is given in Humphreys' recent book on the "Physics of the Air." The character of the effects depends on the size and number of the droplets, their situation relative to the direction of the sun's rays and the position of the observer. Of these the best known are the coronas seen round the sun or moon when viewed through thin clouds, and the familiar rainbows with their accompanying supernumeraries. Less well-known but not of less interest are the glories or brocken-bows seen when a bank of fog or cloud is viewed by an observer in

a balloon or on a mountain, looking away from the sun, nearly in the direction of propagation of his rays. Finally we have the very beautiful iridescent clouds, which are occasionally seen some 15° to 30° from the sun. The object of the present paper is two-fold; firstly to consider the explanations of these phenomena that have been so far advanced, critically from the standpoint of the electro-magnetic theory of diffraction, and to point out in what respects they are inadequate or erroneous: secondly, to describe the laboratory studies which have been made by the writer of some novel and hitherto unobserved features exhibited by iridescent clouds, and to consider their relation to the meteorological phenomena. A summary of the results obtained is given at the end of the paper.

2. *Coronas and Iridescent Clouds.*

G. C. Simpson[1] has advanced the very interesting theory that the iridescent borders and irregular patches of colour occasionally shown by thin high clouds some 15° or 30° from the sun or more are only fragments of coronas formed by exceedingly small undercooled droplets of very approximately uniform size. The experimental study of the light-scattering by such small droplets accordingly acquires some importance, and it was with a view to investigating the theory of iridescent clouds and other related phenomena that the present work was undertaken. The following experimental arrangement was adopted which is essentially the same as that used by Mecke.[2] The condensation chamber in which the clouds are observed is a round-bottomed flask, having a diameter of about 18 cm. and containing a little dilute alkali. The flask is connected with two stop-cocks one of which

[1] Q. Jr. Roy. Met. Soc. 38 p 29.

[2] Ann. der Phys. Vol. 61, 62, 65.

opens to the outer air and the other to a vacuum chamber. By momentarily opening the stop-cock connecting the chamber and condensation flask, clouds are formed within the flask by the adiabatic expansion of the air saturated with water vapour. The vacuum chamber is connected with a manometer to note the changes of pressure. By varying this change of pressure, clouds containing particles of any desired size could be obtained. The particles of largest radius obtained were about 8 μ and the smallest about 1·5 μ.

A 1000 c. p. tungsten filament lamp is placed at a considerable distance from the condensing flask, and a beam of light from it focussed by the lens placed in front of it, at the centre of the flask. When the vapours condense, the track of the beam becomes visible as the result of the scattering of the light by the droplets formed. If observations are made in a direction nearly parallel to the direction of light, the eye being placed on the side opposite to that at which the source of light is situated, one can see vivid patches of colour in the track of the beam, red and green being the most prominent tints. If the droplets are large, these coloured patches can only be seen in directions embraced within a small solid angle, but as the size decreases, they become visible at larger and larger angles. The colour of any patch changes as the angle of observation is changed and the track of the beam through the flask shows different colours at the different parts for the same reason, that is, the angle of observation is different at different parts.

Very small particles are obtained with small expansions at a very low pressure. With very thin clouds consisting of small droplets, the coloured patches could be seen even at a large angle (30°—40°). The observations thus lend an experimental support to Simpson's suggestion about the formation of the iridescent clouds.

When the lens is removed from the arrangement described above and the eye is directed to the source of light through the flask, the coronas may be directly observed when a cloud is formed by expansion. In order that the eye may not be unnecessarily dazzled by the direct light, some opaque obstacle—say the end of the finger may be used to screen it from the direct light. This does not in any way interfere with the observation of the corona. It is found that with smallest droplets the central field is coloured, its tint changing with the size of the droplets; only with larger droplets (larger than about 2μ in the case of water) do normal coronas appear in which the central field is white with a slight-reddish brown edge and surrounded by the usual system of coloured rings. Some measurements were made of the angular diameters of the rings, and the results obtained support Mecke's observation—that except in the case of large droplets, the diameters of the rings show anomalies, the size of the particles as calculated from the simple formula for the different rings showing striking differences.

That the assumptions on which the ordinary theory of coronas is based must fail in the case of very small droplets becomes evident on careful consideration. In the usual treatment the droplets are treated as opaque discs and the problem of finding the diffraction effect due to them is handled with the aid of Babinet's principle. Actually, however, the water droplets are transparent, the rays of light which pass through them emerging as a strongly divergent pencil. If the drops are large, it may be assumed that the rays transmitted in a direction nearly parallel to the direction of the axis have negligible effects. This however is not justifiable in the case of very small drops as the rays passing through the drop and those diffracted at its boundaries or reflected at its outer surface then differ but little in path and hence are capable

of interfering with each other. An attempt has been made by Mecke (*l. c.*) to take account of these complications, and he has obtained a somewhat complex equation for the intensity of the light scattered by the drops. But Mecke's treatment is not satisfactory as he uses elementary methods, the applicability of which in the case of droplets not many times larger than the wave length of light is certainly open to question. The problem evidently calls for a rigorous treatment on the basis of the electromagnetic theory. This is given below in Section 5.

3. *Glories or Brocken-bows and their explanation.*

When favourably situated, one may see rings of coloured light around the shadow of his own head, as cast upon a neighbouring fog bank or cloud. These coloured rings or glories as they are called have been explained in Humphreys' book (*l. c.*) as merely coronas due to the particles near the surface of the cloud scattering light reflected from deeper portions of the cloud, in other words, that the effect is of the same nature as the ordinary corona but due to secondary scattering. This explanation of the brocken-bow has been discussed by Richarz[1] and by Obermayer,[2] and that it cannot be accepted as correct is definitely shown by experimental observations made with artificial clouds. Using the arrangement described previously, if the eye of the observer be placed on the same side of the cloud chamber as the source and looking down very nearly along the path of the beam passing through it, a succession of colours is seen along its track through the cloud. These colours also change as the angle of observation is changed, and the smaller the particles, the greater is the angle from which they can be

[1] Met. Zeit., 1908, 12 and 14

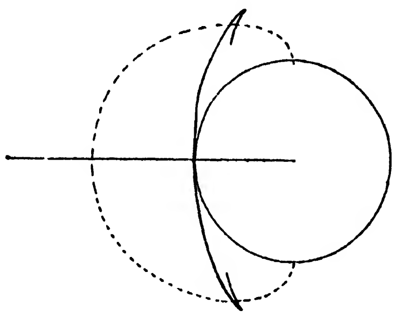
[2] Met. Zeit., 1912.

seen. The complete system of rings is obtained on illuminating the cloud with a beam of sun light and may be viewed in a perpendicular direction with the aid of a plane sheet of glass held at 45° in front of the cloud chamber, so that the observer's head does not screen the cloud chamber from the illuminating pencil. The observations prove clearly that the phenomenon under discussion is shown by every portion of the cloud and therefore really arises from *primary* scattering by the droplets of water.

That the glories or brocken-bows arise in a way which is quite different from that of the ordinary transmission-coronas is proved by the fact that the sequence of colours in the brocken-bows and in the transmission-coronas due to cloud particles of the same size are far from being identical. The normal corona due to large drops shows a central white field with a brownish red margin surrounded by the familiar coloured rings, but in the brocken-bows, the arrangement is different and varies somewhat with the size of the drop. It is sometimes found that just round the central spot (which is the image of the source of light reflected from the first surface of the observing flask) there is a distinct minimum of intensity exhibiting colour, then the intensity increases, the colour being greenish white, bordered by brownish red edge, and then follows the usual succession of coloured rings as in the coronas. It is sometimes found that round the central spot there is a clear maximum and then a belt of minimum intensity and then again a maximum, in other words there is an oscillatory distribution of intensity in the central field. Sometimes it is also observed that in the central field of the brocken-bow only red and green rings or belts are present in different intensities, whitish yellow colour being totally absent, while in the corresponding coronal rings, the central field is yellowish white or nearly without colour.

In order to understand how the glories are formed we have to consider the light which travels back towards the source from the droplets. This arises in two ways, (1) by reflection from the front surface of the droplets, (2) by two reflections and one internal reflection. When a plane wave falls on the spherical particles and is reflected back at its external surface, the reflected wave-front is strongly divergent and as a result, it merely adds a little to the general illumination of the field and does not give rise to any notable diffraction effect But wave-front (2) formed by internal reflection is not so divergent as (1) and is limited by a cusped-edge, at which it is doubled back. See Fig. 1, in which wave-

Fig. 1



front (1) is indicated by dotted lines and wave-front (2) by heavy lines. When the droplet is small, the path differences the between back and front of the wave near the cusped edge are very small. Hence we may without appreciable error consider the wave-front to be a simple spherical cap of appropriate radius. As a sufficient approximation, we may assume the centre of this spherical cap to be the image of a point, placed on the axis at an infinite distance, produced by two refractions and one reflection. We have now to find in directions making a small angle with the axis back towards the direction of the primary source the aggregate effect of this wave

cap. The problem now is the same as the diffraction produced by a small circular opening in a screen on which light is propagated in spherical waves from a point source. We take as the axis of symmetry the line drawn from the source to the centre of the opening and it is required to find the intensity of illumination at any point P of a plane screen parallel to the plane of the opening and at a distance from the latter. Consider now the position of the wave front of radius a which fills the orifice. Let ζ be the distance of P from the axis of symmetry and b the distance of the screen from tne nearest point or pole of the spherical wave of radius a and then using the usual wave equation, the intensity of illumination at P[1] is proportional to

$$M^2=\left(\frac{2}{y}\right)^2(U_1{}^2+U_2{}^2).$$

U_1 and U_2 are calculated by means of Tables of Bessel Functions, where

$$U_1=\Sigma(-1)^n\left(\frac{y}{z}\right)^{2n+1}J_{2n+1}(z)$$

$$U_2=\Sigma(-1)^n\left(\frac{y}{z}\right)^{2n+2}J_{2n+2}(z)$$

$$\tfrac{1}{2}\psi=\frac{2\pi}{\lambda}\,\frac{a+b}{2ab}\,\rho^2 \quad \text{and} \quad \frac{2\pi}{\lambda}\,\frac{\zeta}{b}\,\rho=x$$

$\frac{1}{2}\psi=\frac{1}{2}y$ and $x=z$, where $\rho=r$

$r=$ radius of the orifice.

The above series for U_1 and U_2 are used if $z>y$ but in case when $y>z$

$$U_1+V_1=\sin\tfrac{1}{2}\left(y+\frac{z^2}{y}\right)$$

$$-U_2+V_0=\cos\tfrac{1}{2}\left(y+\frac{z^2}{y}\right)$$

[1] For a detailed mathematical treatment of the problem see Gray and Mathews, Chapter XIV, "Diffraction of Light." Also Lommel. Abh. D. K. Bayer, Akad. D. Wissench. XV, 1886.

where

$$V_0 = \Sigma(-1)^n \left(\frac{z}{y}\right)^{2n} J_{2n}(z)$$

$$V_1 = \Sigma(-1)^n \left(\frac{z}{y}\right)^{2n+1} J_{2n+1}(z)$$

The maxima and minima values of M^2 are those for which

$$\frac{\partial M^2}{\partial z} = 0$$

The solution clearly gives that the maxima and minima are obtained when either

$$J_1(z) = 0, \text{ or } U_2 = 0$$

Curves for M_2 as ordinate, and Z as abcissæ are drawn for

$y = \pi$, 4π, 5π, and 9π (Figs. 2, 3, 4, 5)

Fig. 2

$y = \pi$

$100M^2$

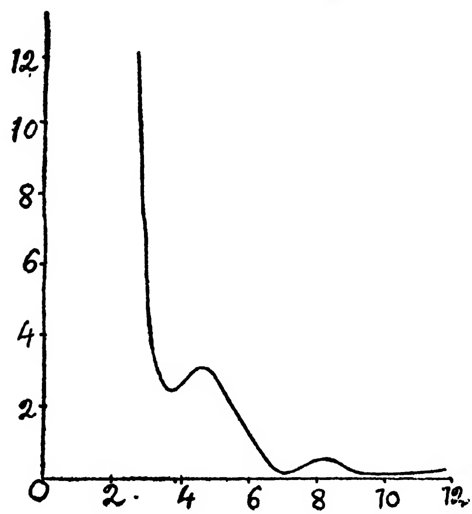


Fig. 3

$y=4\pi$

$100M^2$

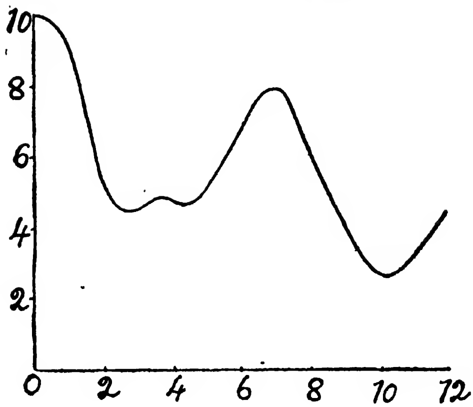


Fig. 4

$y=5\pi$

$100M^2$

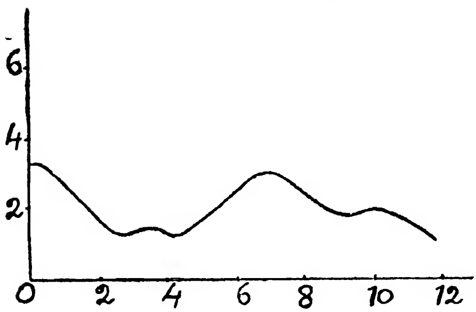
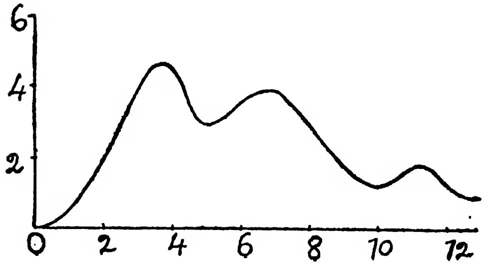


Fig. 5

$y=9\pi$

$1000M^2$



Curves for $y=\pi$, 5π, and 9π as given in Gray and Mathews (l.c.) and for $y=4\pi$ are drawn, the last showing $M^2=0$ when $z=0$.

From the above expressions it is easily seen that

$$M^2=0, \text{ when } y=4n\pi$$

where n may be 1, 2, 3...........etc.

The curves clearly show the peculiarities of the brocken-bow mentioned before. Curves for $y=5\pi$ and 9π show in the central field a belt of minimum intensity round the central spot which is clearly a maximum, showing the oscillatory nature of the intensity in the central field. In the curve $y=4\pi$ (also as it would be in the case of 8π, 12π and so), the central spot is of minimum intensity which only increases from $z=0$ to some higher values of z. These two important peculiarities observed in the central field are clearly explained, the outer rings of the brocken there showing clear maxima and minima as appears from the curves.[1] Some measurements were made of the position of maximum and minimum in the glory—rings and the results are tabulated below. Considering the experimental difficulties and the assumptions in the theory the results agree fairly well with the observations.

BENZOL.

Position of the maximum.

	Observed.	Calculated.	Observed.	Calculated.
4π	5°·4	5°·8	10°·2	10°·8
5π	9°·2	8°·8	11°·5	18°·2
6π	8°	7°·2	11°·1	10° 5

[1] For detailed description of the curves, see Gray & Mathews (loc. cit) and also Lommel, Loc. cit.

WATER.

Position of the maximum.

y	Observed.	Calculated.	Observed.	Calculated.
$2 \cdot 5\pi$	10·1	9·5	...	...
$3 \cdot 5\pi$	7·4	6·9	11·8	12·3
4π	6 5	5·9	9·8	10·5

Intensities and Polarisation of Scattered Light.

Using a lens to condense the beam of light traversing the flask as in the observation of iridescent clouds, it is noticed that when the eye is moved from $\theta = 180°$ (the direction of the transmitted light) to $\theta = 0°$ (i.e., back towards the source), the intensity of the light at first fluctuates forming coronas as described before, and then becomes very small some-where about 90°. It then increases again at first slowly and then very rapidly at about $\theta = 44°$, which is near the position demanded by the ordinary theory for the primary bow. Particles of size 8μ were produced within the condensing chamber yet no trace of colour could be seen in this direction, This position is indicated by a short maximum. Evidently particles larger than 8μ are responsibile for the colours of the rainbow. The intensity of light decreases slightly to increase again as θ is still further decreased ; the exact angles are however difficult to indicate and also the intensity does not fluctuate so largly as on the other side. But a slight fluctuation could be noticed. This suggests the formation of supernumeraries. With still further decrease of θ, the glories are formed which are described before. These observations of the sharp maximum at

about 44° are also consistent with the ordinary theory as deduced by Airy, Pernter, Mobius,[1] and others from which it follows that the rainbow bands produced by very small droplets are not only broad, but also feeble, and as their colours necessarily are faint, they frequently are not distinguished, the bow appearing merely white.[2]

According to the ordinary theory of the rainbow as deduced from wave principles, the supernumeraries should appear, the distance between the maxima decreasing with the increase in size of the droplets. It should be remarked however, that in the solution of the problem it is assumed that the wavefronts are necessarily unlimited or infinite—compared with the wave length of light—on both sides of the inflexion point. This is only valid in the case of very large drops ; but as soon as we come to sizes not excessively large in comparison with the wave length of light, the assumed extended character of the wavefronts has no counterpart in reality and Airy's method of treating the problem ceases to be justifiable and a stricter investigation is called for. The theoretical treatment of this problem will be taken up in section 5.

The coronas or iridescent clouds show no trace of polarisation, and indeed the scattered light from $\theta=180°$ to $\theta=45°$ (nearly) is practically unpolarised. At about $\theta=45°$ the light is polarised, the intensity of $\|^{l}$ component being greater than that of the $\perp^{r}$ one.[3] With further decrease in θ the intensity of $\|^{l}$ component decreases and that

[1] Pernter—Meteorologische Optik.
Mobius—Ann. der Phys., 1910 and 1913.

[2] Humphreys (*l. c*), page 477.

[3] The $\|^{l}$ component is the component having the vibrations in the plane containing the direction of the incident ray and the direction of observation. The vibration is perpendicular to the direction of observation. The $\perp^{r}$ component, has its vibration perpendicular to the above plane. The vibration is also perpendicular to the direction of observation.

of the other increases, and we reach a neutral point at which the two intensities are equal. The two components are differently coloured, generally red and green, the colour also changing as θ is changed. Beyond this the ratio of the intensity of the two components shows an oscillation with varying θ, the oscillations being most rapid, with the larger particles (3μ or more). With the smaller drops in passing through the smaller values of θ, the colour appears sooner in the two components than with larger drops, otherwise the general appearance is the same. The polarisation of the scattered light is thus practically confined within the region $\theta=0^\circ$ to 45° and its oscillatory character is quite prominent. The difference of colour of the two components is perceived at the largest value of θ within this range when the drops are smallest.

Rigorous Treatment of Diffraction Problem on the Electromagnetic Theory.

Let us suppose that a beam of unpolarised light fails on a spherical obstacle with its centre as the origin and also let the light travel in the negative direction along the axis of z. Suppose we confine our observation to a horizontal plane (*i.e.* plane containing Z, X) at a distance r from the centre, and making an angle θ with the incident beam. If X, Y and Z denote the electric forces parallel to the three axes in the scattered wave, then the vertical component of the scattered light is denoted by Y and the horizontal one by $\frac{xZ-xX}{r}$. Love's solution as corrected and modified by the late Lord Rayleigh[1] gives the following expressions for the two components.

[1] Proc. Roy. Soc, Vol. 84, Ser. A, 1010.

$$Y = \sum_{n=1}^{n=\infty} (-1)^{n+1} \frac{2n+1}{n(n+1)} [M_n\{\mu P_n' - (n+1)\, n \cdot P_n\} + N_n P_n'] \frac{e^{ik(ct-r)}}{kr}$$

$$\frac{xZ - zX}{\gamma} = \sum_{n=1}^{n=\infty} (-1)^{n+1} \frac{2n+1}{n(n+1)} [N_n\{\mu P_n' - n(n+1) P_n\} + M_n P_n'] \frac{e^{ik(ct-r)}}{kr}$$

In these equations, $k = \frac{2\pi}{\lambda}$, λ being the wave length of the incident light, $\mu = \cos\theta$ and P_n or $P_n(\mu)$ is a zonal harmonic of degree n whose axis is the axis of z, Mod Y and Mod $\frac{xZ - zX}{r}$ give the amplitudes of the two components and their squares give the intensities. M_n and N_n are functions of the size and optical properties of the spherical particles. The complete expression for N_n is

$$\frac{K\psi_{n-1}(\eta) - \left\{(K-1)\frac{n}{2n+1} + \frac{\psi_{n-1}(\eta')}{\psi_n(\eta')}\right\}\psi_n(\eta)}{-KE_{n-1}(\eta) + \left\{K-1\left(\frac{n}{2n+1}\right) + \frac{\psi_{n-1}(\eta')}{\psi_n(\eta')}\right\}E_n(\eta)}$$

and for M_n

$$\frac{\psi_{n-1}(\eta) - \frac{\psi_{n-1}(\eta')}{\psi_n(\eta')}\psi_n(\eta)}{-E_{n-1}(\eta) + \frac{\psi_{n-1}(\eta')}{\psi_n(\eta')}E_n(\eta)}$$

The expression for M_n is obtained by substituting μ the magnetic permeability instead of K. In optical problems we may take $K=1$ so that the expression for M_n stands as above.

K is the dielectric constant of the material composing the spherical particle, that of the surrounding medium being supposed equal to unity. K may be substituted

for m^2 where m is the refractive index of the material composing the spheres, relatively to the surrounding madium and $\eta'=m\eta$

$$\psi_n=(-1)^n\ 1\cdot3\cdot5\cdot7\cdot9\ldots\ldots\ldots(2n+1)\left(\frac{1}{\eta}\frac{d}{d\eta}\right)^n\frac{\sin\eta}{\eta}$$

and

$$E_n=(-1)^n\ \cdot1\cdot3\cdot5\cdot7\cdot9\ldots\ldots\ldots(2n+1)\left(\frac{1}{\eta}\frac{d}{d\eta}\right)^n\frac{e^{-i\eta}}{\eta}$$

so that $E_n(\eta)=\Psi_n(\eta)-i\,\psi_n(\eta)$ where real and imaginary par's are separated.

In the case of the water droplets m is taken to be $\frac{4}{3}$ or 1·3333, the surrounding medium being air, $K=\left(\frac{4}{3}\right)^2$

It appears from the above expressions that arithmetical calculations with higher value of η or (ka) are very heavy and tedious, but in order to have an idea about the intensity distribution, calculations have been made with $ka=12$, a very tedious piece of work so far as the arithmetical computations are concerned.

In order to calculate the values of $E_n(\eta)$ the sequence formula was used

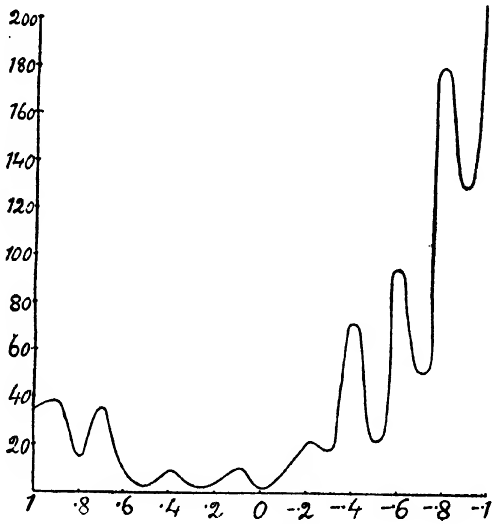
$$E_{n+1}=\frac{(2n+1)\ (2n+3)}{\eta^2}\ [E_n-E_{n-1}]$$

starting from E_0 and E_1. This method is satisfactory as regards the real part of $E_n(\eta)$ but as the imaginary part tends to equality, any error that may creep in is multiplied at the next step by a large factor $(2n+1)(2n+3)$. This difficulty may be overcome in the following manner when the convergence is good. We may calculate the value of ψ_{25} and ψ_{26} by a straight-forward method very accurately. Having obtained them, we may then use the sequence formula in a reverse direction to find the lower values without any loss of accuracy.

The values of Ψ and ψ were calculated for $ka=12$ and tabulated in table I at the end of the paper. The logarithmic values of M_n and N_n are tabulated in table II.

The quantities $\frac{2n+1}{n(n+1)}\mu P_n'-(2n+1)P_n$ or B_n and $\frac{2n+1}{n(n+1)}$ P_n' or A_n are functions of η and μ. Their logarithmic values for $\mu=\cos\theta=0,\cdot1,\cdot2,\cdot3,\cdot4,\cdot5,\cdot6,\cdot7,\cdot8,\ 95,\cdot9,\cdot95,\ 1$, are tabulated and the signs of A_n and B_n have to be changed properly in obtaining arithmetical values when θ is negative.[1]

Fig. 6



|| component

Curves are drawn in Figs. 6 and 7, representing intensity or

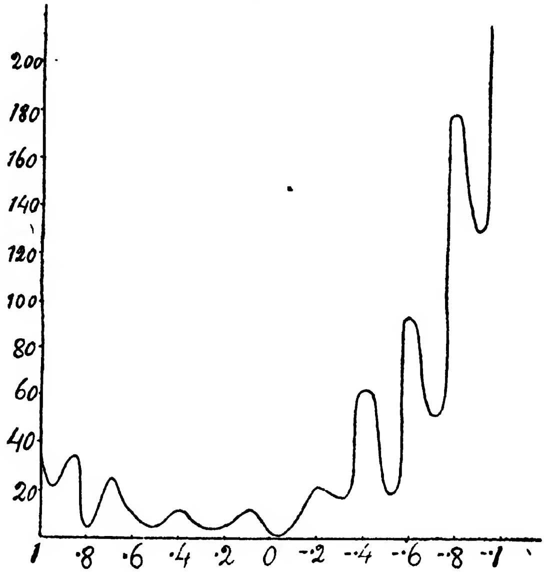
$$(\text{Mod } Y)^2 \text{ and } \left(\text{Mod } \frac{xZ-zX}{r}\right)^2$$

Coronas: It will be seen that in the transmitted direction, the oscillatory character of the intensity of the

[1] Proc. Roy Soc., Vol. 84, 1910.

scattered light is most marked, showing sharp maxima and minima. Clearly these are the positions of the

Fig. 7



⊥ component

normal coronas, described before. According to the ordinary theory, the positions of the maxima and minima are given by the relation $\mathrm{Sin}\theta=\frac{m}{\pi}\frac{\lambda}{r}$. Where these numerical values for the positions of the maxima and minima and compared with those from the rigorous theory (in the case calculated), they are found to differ by 5° to 6°. This discrepancy is to be expected as we have seen before. It is of interest to compare those positions in the case of perfectly conducting particles. Arithmetical calculations for such particles ($ka=9$ and 10) have been made by Proudman, Doodson and Kennedy, (Phil. Trans., A, Vol. 217) on the rigorous electromagnetic theory. The positions for maxima and minima are taken from their paper and when compared with the ordinary theory, are found to agree very well within a few per cent.

Intensity of the scattered light :—When θ decreases, the resultant intensity decreases rapidly till at about 90° it is practically nothing, and sharply increases at 44°, which is near the position of the primary bow. With still further decrease in θ, the intensity fluctuates, showing supernumeraries. Ordinary theory gives the position of the supernumeraries for various sizes of the particle (see Humphreys' book page 476). But for the size we have calculated $ka=12$, the position of the next maximum will be about 8° to 9° from the position of the primary bow, but according to the rigorous calculation it is about 14°-15° from that position. No such sharp maximum as demanded by the ordinary theory at such a distance could be observed. The intensity of the light slightly diminishes only to rise again. Accurate measurements of the positions of the next maximum could not be made, but the approximate position agrees better with the rigorous theory than with the ordinary one, thus showing the limitations of Airy's theory.

Polarisation of the scattered light :—From the graph drawn, it is found that light is not polarised in the direction of transmission nor is it polarised in the direction $\theta=180°$ to 45° (about). At $\theta=45°$ the polarisation can be detected and the best positions for detecting it lie within the region $\theta=0°$ to 45°. This includes the region where the brocken-bow is formed and the rain-bow which is also polarised. These facts agree with what is actually observed experimentally. As a contrast the graphs for conducting particles with $ka=9$ and 10 drawn by Proudmann, Doodson and Kennedy[1] may be referred to. It is found there, that light is only polarised in the directions where the normal coronas are formed and is unpolarised in the directions towards the source. (Just the opposite effect is observed with dielectric particles).

[1] Loc. cit.

Summary and Conclusion.

The paper describes experimental and theoretical work dealing with the following optical effects shown by clouds of small liquid droplets (1) coronas (2) glories or brocken-bows (3) white rainbows and (4) polarisation of scattered light.

The following are the principal results.

(1) Simpson's theory that iridescent clouds seen at large angles from the sun are really fragments of unusually large coronas formed by exceedingly small droplets receives an experimental verification.

(2) The glories or brocken-bows seen when a bank of cloud is viewed by reflected light are shown to be an independent phenomenon due to primary scattering of the sun's rays by the droplets and are not coronas due to secondarily scattered light as has been suggested by some previous writers. They are experimentally found to possess a character different from that of coronas. It is shown that they are due to the integrated effect of the whole wave-front having approximately the form of a spherical cap bounded by a cusped edge emerging after internal reflection at the rear surface of the droplets and the mathematical theory is worked out and shown to be in agreement with experimental results.

(3) The light scattered back nearly in the direction of the source within the region $\theta = 0°$ to $\theta = 45°$ shows remarkable alternations of its state of polarisation in different directions.

(4) A rigorous calculation on the basis of the electromagnetic theory shows that the elementary theory of coronas is considerably at fault in the case of very small drops and also that Airy's theory of supernumerary rainbows ceases to be applicable in the case of very small drops. It also explains the effects observed under (3). The theoretical intensity curves show at a glance the relation

between the various phenomena and the rudimentary development of the white rainbow even in the case of such small particles as 1 μ in radius.

The investigation was carried out in the Palit Laboratory of Physics at the University College of Science, and the author is indebted to Prof. C. V. Raman for his unfailing interest and suggestions during the progress of the work, and especially for his encouragement during the period of tedious arithmetical work involved in the preparation of the paper.

Calcutta, the 23rd July, 1922.

TABLE I

n	Ψ	ψ
0	·070325	—·044708
1	—·0097117	—·018512
2	—·0083374	·0027290
3	·00033404	·0051628
4	·0037938	·0010647
5	·0023783	—·0028174
6	—·0014056	—·0038550
7	—·0051239	—·0014052
8	—·0065843	·0043382
9	—·0032758	·012882
10	·0091670	·023669
11	·041736	·036181
12	·13005	·049981
13	·41397	·064698
14	1·5437	·080034
15	7·0485	·095747
16	39·100	·111654
17	257·01	·127609
18	1959·2	·143501

TABLE II

n	M_n	N_n
1	$\bar{1}\cdot67700-\bar{1}\cdot81631i$	$\bar{1}\cdot69819-\bar{1}\cdot67164i$
2	$\bar{1}\cdot69021-\bar{1}\cdot60272i$	$\bar{1}\cdot68343-\bar{1}\cdot80029i$
3	$\bar{1}\cdot69040-\bar{1}\cdot77697i$	$\bar{1}\cdot67032-\bar{1}\cdot51066i$
4	$\bar{1}\cdot61610-\bar{1}\cdot33923i$	$\bar{1}\cdot69846-\bar{1}\cdot67673i$
5	$\bar{1}\cdot67701-\bar{1}\cdot53764i$	$\bar{1}\cdot59653-\bar{1}\cdot28636i$
6	$\bar{1}\cdot56442-\bar{1}\cdot20463i$	$\bar{1}\cdot53355-\bar{1}\cdot13009i$
7	$\bar{2}\cdot90520-\bar{3}\cdot81322i$	$\bar{1}\cdot45291-\bar{2}\ 94594i$
8	$\bar{1}\cdot30900-\bar{2}\cdot63725i$	$\bar{2}\cdot50959-\bar{3}\cdot01964i$
9	$-\bar{1}\cdot33569-\bar{2}\cdot69340i$	$-\bar{1}\cdot41525-\bar{2}\cdot86341i$
10	$-\bar{1}\cdot69896-\bar{1}\cdot70228i$	$-\bar{1}\cdot55601-\bar{1}\cdot45966i$
11	$-\bar{1}\cdot69838-\bar{1}\cdot67595i$	$-\bar{1}\cdot69888-\bar{1}\cdot70788i$
12	$-\bar{1}\cdot69868-\bar{1}\cdot71434i$	$-\bar{1}\cdot49990-\bar{1}\cdot94811i$
13	$\bar{1}\cdot63080-\bar{1}\cdot38105i$	$\bar{1}\cdot53605-\bar{1}\cdot13602i$
14	$\bar{2}\cdot65045-\bar{3}\cdot30165i$	$\bar{2}\cdot75577-\bar{3}\cdot51296i$
15	$\bar{3}\cdot71746-\bar{5}\cdot57494i$	$\bar{2}\cdot00150-\bar{4}\cdot00300i$
16	$\bar{4}\cdot92540-\bar{7}\cdot85000i$	$\bar{3}\cdot85100-\bar{5}\cdot78216i$
17	$\bar{4}\cdot03062-$	$\bar{4}\cdot39923$
18	$\bar{5}\cdot09800-$	$\bar{5}\cdot52403$

TABLE for $(\text{Mod } Y)^2$ and $\left(\text{Mod } \frac{xZ-zX}{r}\right)^2$

$\mu=\cos\theta$	$(\text{Mod } Y)^2$	$\left(\text{Mod } \frac{xZ-zX}{r}\right)^2$
1·00	34·1	34·1
0·95	21·2	36·2
0·90	28·9	38·3
0·85	34·3	28·6
0·80	4·8	15·2
0·70	24·7	34·9
0·60	10·1	9·5
0·50	4·8	2·9
0·40	11·2	9·1
0·30	4·4	3·1
0·20	4·9	4·6
0·10	11·2	10·4
0·00	2·1	2·0
−0·10	6·3	7·1
−0·20	20·1	20·3
−0·30	17·5	18·1
−0·40	66·6	68·4
−0·50	20·2	21·1
−0·60	94·6	95·1
−0·70	52·0	50·7
−0·80	179·9	181·4
−0·85	163·6	164·8
−0·90	130·1	132·1
−0·95	280·6	282·1
−1·00	848·6	848·6

IV. On the Chronographic Determination of Acceleration of Gravity

By

Durgadas Banerji, M.Sc.,

Demonstrator in Physics,

University College of Science, Calcutta.

Introduction

In his well-known investigations with the phonodeik, D. C. Miller[1] used a chronographic fork giving a very brief flash of light through slits attached to its prongs once in each oscillation which was registered on the moving photographic film and served as a standard against which the frequencies of sounds recorded on the film could be determined. A similar device has been employed for oscillographic determination of frequencies in instruments constructed by the General Electrical Company.[2] The merit of the arrangement lies in the sharpness of the lines representing the successive flashes on the film which permits the measurement of their relative positions to a high degree of accuracy and so surpasses, for all quantitative purposes, the method of recording the vibrations of a tuning fork on a smoked paper ordinarily used in chronographic work. Various types of instrument[3] have been developed using the chronographic or fall plate method for the determination of the acceleration of gravity, but all of them

[1] D. C. Miller, " Science of Musical Sounds. Macmillan, 1914.

[2] Journal of the Franklin Institute, 1922.

[3] Cf. Brown's or Edelmann's apparatus, School Science and Mathematics VIII, p. 387, 1908.

suffer from the common defects—in that (i) the resistance offered by the smoked plate to the style of the recording fork is uncertain and (ii) the computation of distances of fall from the wavy nature of the records is liable to much uncertainty—and so there are, in fact, serious drawbacks in all quantitative determinations of the constant from measurements of time and the corresponding distances of fall.

In the following investigation the acceleration of gravity was determined by dropping a photographic plate on which chronographic flashes were recorded by an electrically-maintained tuning fork as in D. C. Miller's work, the frequency of the fork being simultaneously determined by direct comparison against a standard clock by the method of the sub-synchronous pendulum[4] developed at Calcutta.

Theory of the Method.

The vertical distances of fall of a body from the point of rest being represented by S_1, S_2, S_3......and the corresponding times of fall by t, $t+T$, $t+2T$...... where T is a constant such as the period of a fork, the following equations connect the distances of fall and time with the initial velocity (u), if any, of the body and the acceleration of gravity acceleration at the place.

$$S_1 = ut + \tfrac{1}{2}gt^2 \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1)$$

$$S_2 = u(t+T) + \tfrac{1}{2}g(t+T)^2 \qquad \ldots \qquad \ldots \qquad (2)$$

$$S_3 = u(t+2T) + \tfrac{1}{2}g(t+2T)^2 \qquad \ldots \qquad \ldots \qquad (3)$$

[4] For fuller information *vide* Prof. C. V, Raman and A. Dey, Proc. Roy. Soc., Vol. 95, 1919, also Durgadas Banerji, Proc. I. A. C. S., Vol. VII, Parts III & IV.

Eliminating u and t from the above equations

$$g = \frac{S_3 - S_2 - \overline{S_2 - S_1}}{T^2}$$

$$= n^2[\overline{S_3 - S_2} - \overline{S_3 - S_1}] \quad \dots \quad \dots \quad (4)$$

Where n = frequency of the fork, and $S_3 - S_2$ and $S_2 - S_1$ measure the distances of fall in two successive periods of the fork. The equation (4) being true only for vacuum, a correction term due to frequency is required to be added to (4) for fall under atmospheric or reduced pressure given by

$$+\frac{V \times 0{\cdot}001298}{M(1+0{\cdot}00367t)} \frac{B}{760}$$

where V is the volume and M the mass of the falling body: B and t the pressure and temperature of the air respectively. The viscous drag on the falling body which is independent of pressure but depends on the velocity is not such as to affect the results seriously in the *initial* stages of fall of a heavy body, and the corresponding correction may be left out of account specially in work under reduced pressure.

Apparatus and Experimental Procedure and Results.

Sunlight falling on a rectangular horizontal slit is focussed by a lens on a pair of fine horizontal slits—forming a stroboscopic arrangement—attached to the prongs of a tuning fork (frequency 60/sec.), so that, when the fork is not vibrating, a well-defined horizontal slit of light diverges out from the back of the slits. The stroboscopic slits may be made by fixing in juxtaposition the sharp edges of cut Gillete-razor blades so as to leave a slit of 1/5th mm. width

in metal plates rigidly fixed on to the prongs of the fork. The light passing through the slits is focussed by another lens through a glass-window in a vertically fixed cast-iron pipe in which the photographic plate could be dropped by electromagnetic release in a vertical plane containing the point of focussed light. The pipe is closed below and might be evacuated through an air pump connection below after closing the top with a ground glass lid fitted air-tight by a paste[5] of rubber grease and paraffin. A wooden lid of the pipe fixed below the glass lid holds an electromagnet which could be operated from an outside switch and from which is hung, in bifilar suspension of strings, a rectangular slide constructed so as to hold in vertical position a half sized photographic plate just above the point of focussed light in the same vertical plane with it. The exact verticality of the hanging slide was ensured by attaching counterweights on the back-side of the slide. The slide after being loaded and tested for verticality could be enclosed in a paper cover which is taken out after hanging it in position in the tube.

The procedure adopted in measuring the frequency of the fork accurately is to run in maintained oscillation a 100 cm iron rod pendulum by an electromagnet in series with the interrupted current of the fork. The pendulum being hung in front of the standard laboratory clock, determinations of its frequency could be made each time before and after a record is taken, by noting the coincidence period—for the same phase of motion—of the reflection of a bright source from the mercury bob of the clock pendulum and the shadow of the sub-synchronous pendulum rod cast on a tissue paper scale fixed so as to have the two amplitudes on it equal. The reflection of the slit of light being focussed on the tissue paper by a lens,

[5] A suitable paste is made by mixing together 20 parts of India rubber, 10 parts of grease and 4 parts of hard paraffin over a slow fire.

computations of the coincidence period could be made to within a fifth of a second. Calculations of the frequency of the fork are made from the relation $n = m\mathrm{N}$ where m is the constant frequency ratio (an even integer), and N the calculated frequency of the sub-synchronous pendulum estimated to have an accuracy of 100,000.

In the measurement of the distances of the sharp edges of the records on the Adam Hilger comparator microscope reading to a thousandth part of a mm., the possible sources of error were the discrepancies due to random and systematic errors of setting the cross wire on the edges of the mark and were sensibly minimised by taking a large number of settings on parallel lines near enough and taking the mean. The frilling of gelatine on the photographic plate was not such as to affect the readings to the third place of decimals.

TABLE I.

Serial No.	n^2	$s_3 + s_1 - 2s_2 \times 10^4$	Calculated apparent g.	Buoyancy correction.	Corrected g.	Remarks.
1	3027·661	3232	978·54	0·23	978·77	
2	3027·568	3232	978·51	0·23	978·74	
3	3027·661	3232	978·54	0·23	978·77	Plates under atmospheric pressure.
4	3026·415	3233	978·44	0·23	978·67	
5	3026·601	3233	978·50	0·23	978·73	
6	2950·858	3309	978·42	0·23	978·65	
7	2919·782	3352	978·71	0·05	978·76	
8	2919·249	3353	978·72	0·05	978·77	Plates under pressures 15 to 20 cms.
9	2920·754	3351	978·74	0·05	978·79	
10	2919·782	3352	978·71	0·06	978·77	

Table I gives a few typical measurements from plates for fall with air under the atmospheric pressure and when

reduced to 15-20 cms of mercury by a hand pump. The accepted value of g for Calcutta being 978·76, the results obtained are fairly correct to the first place of decimals. It will be noted that though the buoyancy correction has been reduced considerably under diminished pressure, the viscous drag on the falling plate is not such as to affect the results to the first place of decimals, in pressures of air considerably reduced. It is hoped to continue the work in much higher stages of vacuum when a suitable opportunity arises.

In conclusion, I have to express my best thanks to Prof. C. V. Raman, for the great interest he took in the work.

University College of Science,
Calcutta.

V. On the Oscillations of Spheroidal Drops and the Phenomena of the Spheroidal State.

BY

RAJENDRA NATH GHOSH, M.Sc.,

Lecturer in Physics, University of Allahabad.

[Plate I]

CONTENTS

PART I.—THE OSCILLATIONS OF SPHEROIDAL DROPS

1. *Introduction.*

It is well known that when the upper surface of a vibrating plate is covered with a layer of mobile liquid, the surface usually presents a beautifully crispated appearance. A full account of experiments on this

subject was given by Faraday in the Philosophical Transactions of the Royal Society, 1831. Another series of beautiful forms are observed when, instead of a layer of liquid, we take a drop which does not wet the surface and rests on it in the form of a spheroid. These seem to have been first observed by C. W. Batdorf.[1] In the present note an account is given of the quantitative study of the vibrating drops. The natural frequency of oscillation has been investigated, and it has been found that the amplitude of motion must be taken into consideration in deducing the frequency. The relation between the vertical motion of the support and the horizontal oscillation of the drop has been determined. It has been found that the maintenance falls under the class of those under forces of double frequency as in the cases discussed by Faraday where the motion of the liquid surface is vertical.

2. *Experimental Details.*

In order to study the oscillation of the drops Prof. C. V. Raman's Motor Vibrator[2] was used with a slight modification. The rocking lever was made to vibrate in a vertical direction. Attached to this was a watch-glass which was capable of sliding along the rod and of being fixed in any desired position; large variations of its amplitude of motion could be made by shifting the crank pin along the slot (see p. 447, Physical Review, Nov. 1919), and finer adjustments of amplitude were made by moving the watch-glass along the vibrating lever. The frequency of oscillation was kept constant by a sliding rheostat, and a stroboscopic fork. The frequency was kept constant at about 60 oscillations per second, and clean mercury drops were used.

[1] C. W. Batdorf, Physical Review, Dec. 1912.

[2] Prof. C. V. Raman, Physical Review, Nov. 1919.

In order to measure the diameter and the amplitude of vibration of the drop, the following arrangement was made. Light from an arc lamp was reflected through the bottom of the watch-glass, and by means of a right-angled prism, an enlarged picture was obtained on a white screen by a good lens. This arrangement is suitable for showing the phenomena to a large audience, and for photography.

When the support oscillates up and down, the drop takes the form of a peaked star, the number of star-points depending upon the frequency of vertical oscillation, and on the dimensions of the drop. The crests and troughs of the marginal wave succeed each other so quickly, that we cannot detect the different phases of the motion. Hence we observe double the number of star-points actually present. By changing the dimensions of the drop, the number of star-points may be increased, the frequency of vertical oscillation being kept constant.

Plate I shows some of the star-figures obtained. Fig. (1) represents a two-point star, photographed as a four-point one. The circular figure of equilibrium assumes an elliptical shape. The vibrating part of the drop is easily recognised by the half-shaded portion. A figure (not reproduced) shows a three-point star, the shape of the disturbed figure being an equilateral triangle. Fig. (2) shows a four-point star, the disturbed figure being a square. Figs. (3) and (4) shows five- and six-point stars in which the disturbed figures are a pentagon and hexagon respectively.

3. *Determination of the frequency of the star-points.*

To find out whether there was any definite relation between the frequency of vibration of the star-points, and that of the vertical oscillation for different cases, an electromagnetic vibrator was brought in unison with

the vertical oscillation, and its movement and that of the vibrating star-point, were simultaneously photographed as wavy curves. Figs. 5 and 6 show simultaneous records, from which it is evident that the frequency of vibration of the star-point is half that of the vertical oscillation. In this way, when by varying the diameter of the drop the number of star-points was changed, it was observed that the above relation remained true. It was also noticed that the motion of the star-points was not symmetrical and simple. The nodal points also were not places of absolute rest, but a slight regular motion was perceptible.

4. *Calculation of the natural frequency of oscillation of the star-points.*

The motion of the star-points is practically confined to two dimensions, and then the calculation becomes very simple by Rayleigh's method of energy. Let us represent the radius vector at any instant in the form

$$r = a_0 + \Sigma_1^n \; a_n \cos n\phi \qquad \ldots \quad (1)$$

The potential energy due to capillarity from the configuration of equilibrium is given by

$$V = \tfrac{1}{2}\pi T (n^2 - 1) a_n^2 a \qquad \ldots \quad (2)$$

where a=radius of the drop

T=surface tension

n=number of the star points

The velocity potential of the liquid motion is given by

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} = 0$$

$$\therefore \psi = A r^n \cos n\phi \qquad \ldots \quad (3)$$

where A is determined from the condition

$$\frac{\partial\psi}{\partial r}=\frac{\partial r}{\partial t}\text{ when } r=a.$$

The kinetic energy is by Green's theorem

$$T=\tfrac{1}{2}\rho\int\left\{\psi\frac{\partial\psi}{\partial r}\right\}_{r=a} r\,d\phi$$

$$=\tfrac{1}{2}\rho\,A^2\,n\,a^{2n}.$$

On putting the value of A we obtain

$$T=\tfrac{1}{2}\rho\left(\frac{\partial a_n}{\partial t}\right)^2\frac{a^2}{n} \quad \ldots \quad (4)$$

From (2) and (4) the frequency N is calculated and found to be given by

$$4\pi^2\,N^2=\frac{Tn\,(n^2-1)}{\rho a^3} \quad \ldots \quad (5)$$

Formula (5) is only true when the amplitude of motion of the star points is small. Dr. Bohr[1] has given a formula when the amplitude of motion is finite and not infinitesimal. His formula is given by

$$4\pi^2\,N^2=q^2\left\{1-\frac{a_n^2}{16a^2}\cdot\frac{(n^2-1)\,(34n^3-33n^2+50n-18)}{(2n^2+1)\,(2n-1)}\right\} \quad \ldots \quad (6)$$

where $q=\frac{T_n\,(n^2-1)}{\rho a^3}$ approximately.

From (5) and (6) it is evident that the effect of the finite amplitude of motion is to diminish the natural frequency. Table I shows a comparison of the calculated frequencies from formulæ (5) and (6) respectively.

[1] Phil. Trans. Royal Society, Vol. 209.

TABLE I

Mercury Drops

Frequency of the vertical oscillation 60 per second

Number of star-points.	Diameter cm.	Amplitude	Calculated frequency from (5).	Calculated frequency from (6).	Observed frequency.
3	·51	·08	37	29·6	30
4	·71	·07	36	31·0	30
5	1·0	·05	31	29·1	30
6	1·08	·08	36	29·1	30
7	1·32	·07	34	30·0	30
8	1·42	·08	37	30·0	30
9	1·49	·08	41	32·0	30

The frequency of the star-points should be about 30 per second, and from the table it is evident that the calculated values for formula (5) are much greater than the observed values, while formula (6) agrees well with them.

5. *Maintenance of vibration of star-points.*

The up and down motion of the support disturbs the figure of equilibrium of the drop. The simplest disturbed figure from a circle is an ellipse. This means that when there is contraction of the circular boundary in one direction, there is extension in another, and *vice versâ*. As the support is moving upward, let us suppose that the liquid is also moving away from the centre of drop in a horizontal direction. At the instant when the

support is at the extremity of its upward journey, the liquid still moves outward on account of its inertia, stopping when the support is at the extremity of its downward journey. At this instant the restoring force is very large, and as the support moves upward, the liquid moves inward ; this inward motion comes to an end when the support is at the extremity of its downward journey. This outward and inward motion of the liquid shows up as star-points. It is now clear from the nature of the motion of the star-points that the period of oscillation is double that of the support. It is also evident that the motion cannot be symmetrical since the inward motion is to some extent resisted, hence the oscillation curve of the star-point, see Figs. 5 and 6, is asymmetrical. The curve also shows presence of harmonics.

Part II.—The Phenomena of the Spheroidal State

1. *Introduction.*

The phenomena of the spheroidal state of water engrossed the attention of the early physicists whose first object was to establish whether the drop of water touched the surface of the hot plate or not. This being settled, the question arose as to how the drop is enabled to float on a cushion of its own vapour. In 1878, Johnstone Stoney[1] put forward the theory of polarised stress arising from the difference of temperature of the small gap on its two surfaces. His expression for the stress S is given by

$$\frac{Q^2}{PT}$$

where Q represents the heat conducted across the gap per second, P is the vapour pressure, and T the absolute temperature of the gap. Q can be calculated if the width of the gap, its area, and the thermal conductivity are

[1] British Assoc. Report, 1878.

known. Hence if Stoney's theory be correct, the width of the gap at different temperatures of the hot plate must be capable of calculation.[1] With a view to test this, the investigation of the theory of the spheroidal state was undertaken at the suggestion of Dr. Raman. The experiments resulted in establishing a definite relation between the width of the gap and the difference of temperature between the hot plate and the drop, and some interesting facts were also noted concerning the gap and the star-form oscillation of the spheroidal drop.

2. *Experimental arrangements.*

A simple arrangement was made for the observation of the gap. The drop rested on a polished convex plate of silver which was heated by a Bunsen burner, and is anchored by a fine wire held above it, to which the drop sticks without interfering with the phenomena at all. A powerful beam of light from an arc-lamp passes through the narrow gap between the drop and the plate, and a magnified image of the gap is focussed by a camera lens on a screen where its behaviour can be observed. All superfluous light can be cut off by suitable screens placed behind the drop. The temperature of the hot plate was noted by a thermo-couple soldered to it. The temperature of the drop almost remains constant at 95·6°C as has been shown by Batdorf,[2] hence the difference of temperature between the hot plate and the drop was simply obtained by noting the temperature of the hot plate, and subtracting 96°C from it.

In order to study the variation of the width of the gap with temperature, the mass of the drop was kept constant by keeping the size of the image the same on

[1] See also Maxwell's expression for the polarized stress, Phil. Trans., 1879, p. 252.

[2] Physical Review, Dec. 1912.

the screen. Careful measurements of the gap at various temperatures were taken, and a table was drawn up. It was found that the width of the gap was definite at a fixed temperature of the hot plate when the size of the drop was the same. Table I shows the widths of the gap at various temperature differences D.

TABLE I

Size of drop kept constant at $r = 7 \cdot 5$ mm.

No.	h in mm.	D°c.
1	1·25	224
2	1·40	304
3	1·60	490
4	1·50	364
5	1·40	304
6	1·25	224
7	·75	104

In the course of these experiments, it was observed that a fine droplet when thrown upon the hot plate began to jump like an elastic ball. This led to the enquiry if the spheroidal drop actually oscillates up and down in a vertical plane. This point was ascertained by photographing the gap on a moving plate. The various phases of the gap were at once evident from the picture. It was found that the gap was not stationary; but opened and closed periodically. The rate of the periodic opening and closing of the gap depends in a compound manner upon the difference of temperature between the hot plate and drop and the size of the drop. Fig 7 in Plate I illustrates the periodic closing and opening of the gap,

the simultaneous record of a style attached to an electrically maintained fork making 60 vibrations per second indicating the frequency of the motion. Since the hot plate is stationary, the closing and opening of the gap must be due to the vertical oscillation of the drop, the frequency of the latter being the same as that of the closing and opening of the gap. The asymmetrical character of the vertical oscillation of the drop is clear from the figure where the white space (when light passes freely through the gap) has a greater width than the dark space, *i.e.*, phase of obstruction to light.

It was found, however, that there are in fact two types of the spheroidal state. (1) In the first type (100° difference of temperature, more or less), the drop is stationary, no oscillation takes place, nor is any ripple motion observed on the surface of the drop, though convection currents may be visible. (2) At about 200° difference of temperature the drop begins to oscillate up and down, resulting in periodic closing and opening of the gap. Fine ripples appear on the surface, and vigorous motion of the drop sets in; sometimes the drop takes the form of an elongated spheroid, moving bodily from place to place If this motion is stopped the drop takes a star-shaped form. Batdorf has given a complete study of these star-forms. The peaks of these star-forms vibrate in the horizontal plane, but on account of the persistence of vision the drop appears stationary. Stroboscopic observation renders the oscillation apparent. The number of these star-points, *i.e.*, the peaks observed is double the number that is actually present. The number of these star-points depends upon the temperature difference and the size of the drop. It was found that by slightly moving the burner, and thus decreasing the temperature, the peaks begin to decrease in magnitude, and then suddenly another pattern starts up, the size of the drop

remaining almost the same. If we keep the temperature constant, we obtain a series of forms in which the star-points diminish in number, as the size goes on diminishing by escape of vapour from the gap and evaporation. For the success of these experiments, distilled water should be employed, and the plate must always be clean and polished. To start the star-forms, sometimes slight coaxing is necessary. As soon as the body-motion of the drop sets in, it must be concluded that the drop will soon take up the star-form. By employing a shallow plate and a large drop, big star-forms are obtained.

3. *Theory of Spheroidal State.*

A relation between D and h can be obtained on the assumption that the whole quantity of steam formed per second escapes from the gap according to the laws of viscosity, and that the radial velocity of steam is zero at the surface of the hot plate and at the bottom of the drop. With these premises, we have the quantity M of steam escaping per second

$$M = \frac{h^3 P}{6\mu} \rho \qquad \dots \quad (7)$$

where P denotes the excess of presure within the gap over the outside, ρ is the density and μ the viscosity of the steam. The heat conducted across the gap per second is given by

$$Q = \frac{KD}{h} \cdot \pi a^2 \qquad \dots \quad (8)$$

where K represents conductivity, and a the radius of the drop. Hence the quantity of steam formed per second is equal to

$$M = \frac{KD}{hL} \pi a^2 \qquad \dots \quad (9)$$

L = latent heat of steam.

Equating (3) and (5) we have

$$h^1 = D\left[\frac{6\mu K\pi a^2}{LP\rho}\right] \quad \ldots \quad (10)$$

Equation (10) which is dimensionally correct shows that h is proportional to D if the term in the bracket remains constant. For the same size of the drop, the term in the bracket is almost a constant; from the observed values of h and D for a given size of the drop given in Table I, a graph was drawn with h^1 as ordinate and D as abscissa. The curve is found to be accurately a straight line. Hence the difference of temperature between the hot plate and the drop is directly proportional to the fourth power of the width of the gap. The absolute value of h can be calculated from (10) by assuming that the pressure inside is greater than outside, such that P. πa^2 is equal to the weight of the drop. The calculated value of h comes out a little higher, probably on account of the uncertain values of the constants in the circumstances, and moreover there is always evaporation at the surface. At any rate, we can regard equation (10) to be approximately true.

We have seen that the spheroidal drop oscillates vertically; the formation of the star-forms with vibrating peaks can now be explained in terms of the vertical oscillation. Let us recall the behaviour of a drop of mercury resting on a watch-glass made to oscillate vertically discussed in Part I of the paper. The drop takes the form of a star with vibrating peaks. The frequency N of these peaks vibrating in the horizontal plane, must be half that of the vertical oscillation N^1, therefore

$$N = N^1/_2 \quad \ldots \quad (11)$$

and the number n of the peaked points is determined by

$$n^3 - n + \pi^2 N^2 a^3 \rho / T_s = 0 \text{ [1]} \quad \ldots \quad (12)$$

where T is the capillary tension, ρ the density and a the radius of the drop.

[1] Rayleigh's Theory of Sound, Vol. II, p. 355.

Table II has been drawn up from (11) and (12) showing the values of N, a and n the number of peaked points.

TABLE II

Water Drops

No.	N	a cm.	n
1	50	·74	4
2	100	·72	5
3	300	·63	6
4	50	·42	3
5	100	·57	4
6	300	·51	5
7	300	·30	3

It is observed from (1), (2) and (3) that for about the same size of the drop when the frequency of vertical oscillation is increased two times and three times, the number of peaked points increases from four to five and five to six respectively. For the same value of N, when the size of the drop is halved, the number of peaks is also halved, as shown by (3) and (7). The number of peaks is approximately proportional to the size of the drop when N remains the same.

Just as mercury drops are thrown into vibrating star forms by a vertical motion of the support, the same effect occurs in the case of spheroidal drops of water. On account of the vertical oscillation, the drop takes the form of a vibrating star. The frequency of these vibrating peaks being always half that of the vertical oscillation,

the number of the star-points is determined by the size of the drop. For a given size of the drop, when the temperature rises, the rate of vertical oscillation N increases, so also the number of star-points, as shown in Table II, and remarked above in the preceding paragraph. When the temperature remains the same, and the rate of vertical oscillation remains constant, the frequency of the star-points has the same value, but the number of star-points increases with increasing size of the drop. But when the size of the drop is increased to such an extent as to change the rate of vertical oscillation, the resulting change in the number of star-points is a complicated one and cannot be calculated from (12) as N is indeterminate, *i.e.*, the function upon which N depends in the case of spheroidal drops of water, is unknown.

The number of star-points is always an integer. Hence, when it so happens that the size of the drop is such that equation (12) is not satisfied with n an integer, fine ripples appear upon the surface, vigorous motion sets in, till the size changes by evaporation and escape of steam from the gap, and equation (12) is satisfied. Then the star-form begins at once. These ripples and body motion of the drop are also observed in the case of mercury drops oscillating vertically when (12) is not satisfied.

Synopsis and Conclusion.

Part I of the paper discusses the theory of the star-form oscillations of mercury drops resting upon a vertically oscillating glass plate. It is shown that the frequency of horizontal oscillation is half that of the vertical oscillation. The presence of harmonics whose frequency is $\sqrt{2}$ times that of the vertical movement is also noticed. It is shown that the large amplitudes of motion maintained in practice have a very sensible

influence on the natural frequency of the drop and must be allowed for, and that they also result in a distinct asymmetry in the oscillations.

Part II discusses the theory of the spheroidal state. It is shown that there are two types of spheroidal state, one in which the gap between the drop and the hot plate is stationary, and the seconds type in which the gap opens and closes periodically. The second generally obtained when the temperature of the plate is high; the drop in fact executes a vertical oscillation with a frequency which may be a few hundreds per second. If the frequency of vertical oscillation of the drop be approximately twice that of a possible horizontal oscillation, the drop commences oscillating in a star-form, in other cases its surface is merely thrown into ripples. It is shown, that for a given size of the drop, the fourth power of the width of the gap is proportional to the difference of temperature between the plate and the drop. A quantitute explanation of this result is given as due to the viscous flow of the steam outwards through the edge of the gap under the excess presure which sustains the weight of the drop.

The experiments described above, were performed at the Laboratory of the Indian Association for the Cultivation of Science, and the author begs to record his deep thanks to Prof. C. V. Raman for his interest and help during the progress of the work.

MATHEMATICS

Geometrical Construction for the Limiting Centres of a Cubic

BY

Gurudas Bhar.

In my paper on "The Osculating Conic at Infinity" (*Bulletin C. M. S., Vol. XII, No. 4*), I defined a *limiting centre* of a curve as the centre of a conic osculating the curve at infinity and proved the theorem that *the three limiting centres of a cubic with three real asymptotes are collinear.* The following simple geometrical construction for the limiting centres of the cubic will be found interesting. It furnishes at the same time an elegant proof of the above theorem.

Suppose the three real asymptotes BC, AB and AC meet the curve again at P, Q and R respectively. Then P, Q, R are known to be collinear. We shall show that the corresponding limiting centres P′, Q′ and R′ which lie on BC, AB and AC respectively are such that

$$PC=P'B$$

$$QB=Q'A$$

$$RC=R'A.$$

Let the equation of the cubic be as before

$$(y+\alpha_1 x-\beta_1)(y-\alpha_2 x)y+ly+mx+n=0;$$

let A be the origin; BC, AB and AC be the asymptotes

$$y=\alpha_1 x+\beta_1,$$

$$y=\alpha_2 x,$$

and $$y=0$$

respectively. If the co-ordinates of P, Q, R be (X_1, Y_1), (X_2, Y_2), (X_3, Y_3), those of P′, Q′, R′ be (X_1', Y_1'), (X_2', Y_2'), (X_3', Y_3')

and those of A, B, C be (X_a, Y_a), (X_b, Y_b), (X_c, Y_c) we have

$$X_1 = -\frac{l\beta_1 + n}{l\alpha_1 + m},$$

$$Y_1 = \frac{m\beta_1 - n\alpha_1}{l\alpha_1 + m};$$

$$X_2 = -\frac{n}{l\alpha_2 + m},$$

$$Y_2 = -\frac{n\alpha_2}{l\alpha_2 + m};$$

$$X_3 = -\frac{n}{m},$$

$$Y_3 = 0.$$

Also (see my paper on *The Osculating Conic at Infinity*).

$$X_1' = \frac{-\alpha_1\,\beta_1\,(l\alpha_1 + m) + (\alpha_1 - \alpha_2)\,(n\alpha_1 - m\beta_1)}{\alpha_1\,(\alpha_1 - \alpha_2)\,(l\alpha_1 + m)}$$

$$Y_1' = \frac{-\alpha_2\,\beta_1\,(l\alpha_1 + m) + (\alpha_1 - \alpha_2)\,(n\alpha_1 - m\beta_1)}{(\alpha_1 - \alpha_2)\,(l\alpha_1 + m)}$$

$$X_2' = \frac{n\,(\alpha_1 - \alpha_2) - \beta_1\,(l\alpha_2 + m)}{(\alpha_1 - \alpha_2)\,(l\alpha_2 + m)},$$

$$Y_2' = \frac{n\alpha_2\,(\alpha_1 - \alpha_2) - \alpha_2\,\beta_1\,(l\alpha_2 + m)}{(\alpha_1 - \alpha_2)\,(l\alpha_2 + m)};$$

$$X_3' = \frac{n\alpha_1 - m\beta_1}{m\alpha_1},$$

$$Y_3' = 0.$$

Further

$$X_a = 0,$$

$$Y_a = 0;$$

$$X_b = -\frac{\beta_1}{\alpha_1 - \alpha_2},$$

$$Y_b = -\frac{\alpha_2\,\beta_1}{\alpha_1 - \alpha_2};$$

$$X_c = -\frac{\beta_1}{\alpha_1},$$

$$Y_c = 0.$$

Therefore

$$X_c - X_1 = -\frac{\beta_1}{\alpha_1} + \frac{l\beta_1 + n}{l\alpha_1 + m} = \frac{n\alpha_1 - m\beta_1}{\alpha_1 \, (l\alpha_1 + m)} \,,$$

$$X_1' - X_b = \frac{-\alpha_1 \, \beta_1 \, (l\alpha_1 + m) + (\alpha_1 - \alpha_2)\,(n\alpha_1 - m\beta_1)}{\alpha_1 \, (\alpha_1 - \alpha_2)\,(l\alpha_1 + m)} + \frac{\beta_1}{\alpha_1 - \alpha_2}$$

$$= \frac{n\alpha_1 - m\beta_1}{\alpha_1 \, (l\alpha_1 + m)} \,,$$

or $\quad X_c - X_1 = X_1' - X_b \,;$

and $\quad Y_c - Y_1 = \dfrac{n\alpha_1 - m\beta_1}{l\alpha_1 + m} \,,$

$$Y_1' - Y_b = \frac{-\alpha_2 \, \beta_1 \, (l\alpha_1 + m) + (\alpha_1 - \alpha_2)\,(n\alpha_1 - m\beta_1)}{(\alpha_1 - \alpha_2)\,(l\alpha_1 + m)} + \frac{\alpha_2 \, \beta_1}{\alpha_1 - \alpha_2}$$

$$= \frac{n\alpha_1 - m\beta_1}{l\alpha_1 + m} \,,$$

or $\quad Y_c - Y_1 = Y_1' - Y_b \,;$

hence $\quad PC = P'B.$

Again $\quad X_b - X_2 = -\dfrac{\beta_1}{\alpha_1 - \alpha_2} + \dfrac{n}{l\alpha_2 + m}$

$$= \frac{n(\alpha_1 - \alpha_2) - \beta_1 \, (l\alpha_2 + m)}{(\alpha_1 - \alpha_2)\,(l\alpha_2 + m)} \,,$$

$$X_2' - X_a = \frac{n(\alpha_1 - \alpha_2) - \beta_1 \, (l\alpha_2 + m)}{(\alpha_1 - \alpha_2)\,(l\alpha_2 + m)} \,,$$

or $\quad X_b - X_2 = X_2' - X_a ,$

and $\quad Y_b - Y_2 = -\dfrac{\alpha_2 \, \beta_1}{\alpha_1 - \alpha_2} + \dfrac{n\alpha_2}{l\alpha_2 + m}$

$$= \frac{n\alpha_2 \, (\alpha_1 - \alpha_2) - \alpha_2 \, \beta_1 \, (l\alpha_2 + m)}{(\alpha_1 - \alpha_2)\,(l\alpha_2 + m)}$$

$$Y_2' - Y_a = \frac{n\alpha_2 \, (\alpha_1 - \alpha_2) - \alpha_2 \, \beta_1 \, (l\alpha_2 + m)}{(\alpha_1 - \alpha_2)\,(l\alpha_2 + m)}$$

or $\quad Y_b - Y_2 = Y_2' - Y_a$;

therefore $\quad QB = Q'A$.

Finally $X_c - X_3 = -\frac{\beta_1}{\alpha_1} + \frac{n}{m} = \frac{n\alpha_1 - m\beta_1}{m\alpha_1}$,

$$X_3' - X_a = \frac{n\alpha_1 - m\beta_1}{m\alpha_1},$$

or $\quad X_c - X_3 = X_3' - X_a$;

and $\quad Y_c - Y_3 = 0 = Y_3' - Y_a$;

therefore $\quad RC = R'A$.

It follows that the three limiting centres P′, Q′, R′ are collinear, as P, Q, R are collinear.

In conclusion, my best thanks are due to Professor S. Mukhopadhyaya at whose suggestion and under whose guidance this paper was written.

Note on Certain Properties of Legendre Polynominals of the Second Type

BY

K. Basu.

§. 1.

A study of the second type of solution of Legendre's differential equation has recently been made by Prof. Nicholson[1] who had to evoluate the definite integral $\int_{-1}^{1} [Q_n(\mu)]^2 \, d\mu$, in connection with the problem of two conducting disks: the capacity of an electrical condenser of this type depends on an integral involving Q functions. This short note is an attempt to investigate into some of its special features with reference to the analogous behaviour of the function $P_n(\mu)$. The function has been studied by Legendre,[2] Schläfli,[3] Heine,[4] Neumann,[5] Whittakar,[6] Nicholson[7] and others. Prof. Nicholson has established very recently certain important results of which I have made use of the following:—

$$(i) \quad \int_{-1}^{1} Q_{2m}(\mu)\, P_{2n+1}(\mu)\, d\mu = \frac{-2}{(2m-2n-1)\,(2m+2n+2)},$$

$$(ii) \quad \int_{-1}^{1} Q_{2m+1}(\mu)\, P_{2n}(\mu)\, d\mu = \frac{-2}{(2m-2n+1)\,(2m+2n+2)},$$

[1] Phil. Mag. Vol. 43. Jan. 1922.

[2] Calcul Integral, t. II.

[3] Über die zwei Heine'schen Kugelfunktionen (Bern, 1881).

[4] Crelle, Bd. XLII. Handbuch der Kugelfunktionen (Bern. 1878).

[5] Ibid, Bd. XXXVII, p. 24.

[6] Modern Analysis, Chap. XV, Second ed.

[7] Loc. cit.

(iii) $$\int_0^1 Q_{2m}(\mu)\, P_{2n+1}(\mu)\, d\mu = \frac{-1}{(2m-2n-1)(2m+2n+2)}\,.$$

(iv) $$\int_0^1 Q_{2m+1}(\mu)\, P_{2n}(\mu)\, d\mu = \frac{-1}{(2m-2n+1)(2m+2n+2)}\,.$$

(v) $\int_{-1}^{1} P_p(\mu)\, Q_q(\mu)\, d\mu = 0$, in all cases in which p and q are both odd or both even.

(vi) $$Q_{2m}(\mu) = -\sum_{n=0}^{\infty} \frac{(4n+3)\, P_{2n+1}(\mu)}{(2m-2n-1)(2m+2n+2)}\,,$$

(vii) $$Q_{2m+1}(\mu) = -\sum_{n=0}^{\infty} \frac{(4n+1)\, P_{2n}(\mu)}{(2m-2n+1)(2m+2n+2)}\,,$$

the last two formulae are convergent when μ is between ± 1, both exclusive.

(viii) $$Q^r_{2m}(\mu) = -\sum_{n=0}^{\infty} \frac{(4n+3)\, P^r_{2n+1}(\mu)}{(2m-2n-1)(2m+2n+2)}\,,$$

(ix) $$Q^r_{2m+1}(\mu) = -\sum_{n=0}^{\infty} \frac{(4n+1)\, P^r_{2n}(\mu)}{(2m-2n+1)(2m+2n+2)}\,.$$

§. 2.

When n is a positive integer or zero, $Q_n(\mu)$ is defined by :—

$$Q_n(\mu) = \frac{1}{2^{n+1}} \int_{-1}^{1} (1-t^2)^n\, (\mu-t)^{-n-1}\, dt.$$

this gives $$\int_{-1}^{1} Q_n(\mu)\, d\mu = \frac{1}{2^{n+1}} \int_{-1}^{1}\int_{-1}^{1} (1-t^2)^n\, (\mu-t)^{-n-1}\, dt\, d\mu$$

$$= -\frac{1+(-)^{n+1}}{n(n+1)}$$

$$= 0 \text{ or } -2/n(n+1),$$

according as n is even or odd. This was obtained by Nicholson by another method. The recurrence formula

$$(n+1)\ Q_{n+1}\ (\mu)-(2n+1)\ \mu\ Q_n\ (\mu)+nQ_{n-1}(\mu)=0$$

gives $\mu Q_{2m}\ (\mu)=\{(2m+1)\ Q_{2m+1}\ (\mu)+2mQ_{2m-1}\ (\mu)\}/4m+1.$

or $$\int_{-1}^{1}\mu Q_{2m}\ (\mu)\ d\mu=\frac{2m+1}{4m+1}\cdot\frac{-2}{(2m+1)\ (2m+2)}$$

$$+\frac{2m}{4m+1}\cdot\frac{-2}{(2m-1)\ 2m}$$

$$=\frac{-2}{4m+1}\left\{\frac{1}{2m+2}+\frac{1}{2m-1}\right\}$$

$$=-1/(m-1)\ (2m-1)$$

also $$\int_{-1}^{1}\mu Q_{2m+1}\ (\mu)\ d\mu=\frac{2m+2}{4m+3}\cdot\int_{-1}^{1}Q_{2m+2}\ (\mu)\ d\mu$$

$$+\frac{2m+1}{4m+3}\int_{-1}^{1}Q_{2m}\ (\mu)\ d\mu=0.$$

that is $$\int_{-1}^{1}\mu Q_n\ (\mu)\ d\mu=\begin{cases}0, & \text{when } n \text{ is odd}\\ -2/(n+2)\ (n-1), & \text{when } n \text{ is even.}\end{cases}$$

Again $(n+1)\ \mu Q_{n+1}\ (\mu)-(2n+1)\ \mu^2 Q_n\ (\mu)+n\mu\ Q_{n-1}\ (\mu)=0$

$$\therefore (n+1)\int_{-1}^{1}\mu Q_{n+1}\ (\mu)-(2n+1)\int_{-1}^{1}\mu^2 Q_n\ (\mu)\ d\mu$$

$$+n\int\mu Q_{n-1}\ (\mu)\ d\mu=0$$

Suppose $n=2m$,

$$\therefore (2m+1)\int_{-1}^{1}\mu Q_{2m+1}(\mu)-(4m+1)\int_{-1}^{1}\mu^2 Q_{2m}(\mu)\,d\mu$$

$$+2m\int_{-1}^{1}\mu Q_{2m-1}(\mu)\,d\mu=0$$

$$\therefore \int_{-1}^{1}\mu^2 Q_{2m}(\mu)\,d\mu=0.$$

Next suppose $n=2m+1$,

$$\therefore (2m+2)\int_{-1}^{1}\mu Q_{2m+2}(\mu)\,d\mu-(4m+3)\int_{-1}^{1}\mu^2 Q_{2m+1}(\mu)\,d\mu$$

$$+(2m+1)\int_{-1}^{1}\mu Q_{2m}(\mu)=0.$$

$$i.e.\ \int_{-1}^{1}\mu^2 Q_{2m+1}(\mu)\,d\mu=\frac{(2m+2)}{4m+3}\cdot\frac{-2}{(2m+4)\,(2m+1)}$$

$$+\frac{(2m+1)}{4m+3}\cdot\frac{-2}{(2m+2)\,(2m-1)}$$

$$=-\frac{1}{4m+3}\cdot\left\{\frac{2m+2}{(m+2)\,(2m+1)}+\frac{2m+1}{(m+1)\,(2m-1)}\right\}.$$

Secondly take the recurrence formula:—

$$(\mu^2-1)\,Q'_n(\mu)=n\mu Q_n(\mu)-nQ_{n-1}(\mu).$$

Suppose $n=2m$

$$\text{thus} \int_{-1}^{1} (\mu^2-1)\, Q'_{2m}(\mu)\, d\mu = 2m \int_{-1}^{1} \mu Q_{2m}(\mu)\, d\mu - 2m \int_{-1}^{1} Q_{2m-1}(\mu)\, d\mu$$

$$= 2m \cdot \frac{-2}{(2m+2)(2m-1)} - 2m \cdot \frac{-2}{(2m-1)\,2m}$$

$$= \frac{-2m}{2m-1} \left\{ \frac{1}{(m+1)} - \frac{1}{m} \right\}$$

$$= \frac{2m}{m(m+1)(2m-1)} = 2/(m+1)(2m-1)$$

Next, suppose $n=2m+1$,

$$\text{thus} \int_{-1}^{1} (\mu^2-1)\, Q'_{2m+1}(\mu)\, d\mu = (2m+1) \int_{-1}^{1} \mu Q_{2m+1}(\mu)\, d\mu - (2m+1)$$

$$\times \int_{-1}^{1} Q_{2m}(\mu)\, d\mu = 0.$$

$$\text{that is} \int_{-1}^{1} (\mu^2-1)\, Q'_n(\mu)\, d\mu = \begin{cases} 0, & \text{when } n \text{ is odd.} \\ 4/(n+2)(n-1), & \text{when } n \text{ is even} \end{cases}$$

Thirdly the recurrence formula

$$Q'_{n+1}(\mu) - Q'_{n-1}(\mu) = (2n+1)\, Q_n(\mu)$$

gives,

$$Q'_{n-1}(\mu) - Q'_{n-3}(\mu) = (2n-3)\, Q_{n-2}(\mu)$$

$$Q'_{n-3}(\mu) - Q'_{n-5}(\mu) = (2n-7)\, Q_{n-4}(\mu)$$

$$Q'_{n-5}(\mu) - Q'_{n-7}(\mu) = (2n-11)\, Q_{n-6}(\mu)$$

etc. etc.

$$\therefore Q'_{n+1}(\mu) = (2n+1)\, Q_n(\mu) + Q'_{n-1}(\mu)$$

$$= (2n+1)\, Q_n(\mu) + (2n-3)\, Q_{n-2}(\mu) + Q'_{n-3}(\mu)$$

$$= (2n+1)\, Q_n(\mu) + (2n-3)\, Q_{n-2}(\mu) + (2n-7)\, Q_{n-4}(\mu) + Q'_{n-5}(\mu)$$

and so on, until the last term is attained.

When n is even

$$\therefore Q'_{n+1}(\mu)=(2n+1)\,Q_n(\mu)+(2n-3)\,Q_{n-2}(\mu)+\ldots+Q_1'(\mu)$$

When n is odd

$$Q'_{n+1}(\mu)=(2n+1)\,Q_n(\mu)+(2n-3)\,Q_{n-2}(\mu)+\cdots+Q_0'(\mu)$$

Now
$$Q_0(\mu)=\tfrac{1}{2}\log\frac{\mu+1}{\mu-1}$$

$$Q_1(\mu)=\tfrac{1}{2}\,\mu\log\frac{\mu+1}{\mu-1}-1$$

$$\therefore Q_0'(\mu)=\frac{1}{1-\mu^2}.$$

$$Q_1'(\mu)=\tfrac{1}{2}\log\frac{\mu+1}{\mu-1}+\frac{\mu}{1-\mu^2}.$$

Hence
$$Q'_{n+1}(\mu)=\left\{\begin{array}{l}(2n+1)\,Q_n(\mu)+(2n-3)\,Q_{n-1}(\mu)+\ldots\\ \qquad\qquad+\dfrac{1}{1-\mu^2}.\ (n\text{ odd})\\ (2n+1)\,Q_n(\mu)+(2n-3)\,Q_{n-1}(\mu)+\ldots\\ \qquad\qquad+\tfrac{1}{2}\log\dfrac{\mu+1}{\mu-1}+\dfrac{\mu}{1-\mu^2}.\ (n\text{ even})\end{array}\right.$$

§. 3.

$$Q_{2m}(\mu)\,Q_{2r}(\mu)=\sum_{n=0}^{\infty}\frac{(4n+3)\,P_{2n+1}(\mu)}{(2m-2n-1)\,(2m+2n+2)}.$$

$$\sum_{p=0}^{\infty}\frac{(4p+3)\,P_{2p+1}}{(2r-2p-1)\,(2r+2p+2)}.$$

$$\therefore\int_{-1}^{1}Q_{2m}(\mu)\,Q_{2r}(\mu)\,d\mu$$

$$=\sum_{n=0}^{\infty}\frac{(4n+3)^2}{(2m-2n-1)\,(2m+2n+2)\,(2r-2p-1)\,(2r+2p+2)}.$$

$$\int_{-1}^{1}P^2_{2n+1}(\mu)\,d\mu$$

$$=2\sum_{n=0}^{\infty}\frac{4n+3}{(2m-2n-1)\,(2m+2n+2)\,(2r-2p-1)\,(2r+2p+2)}$$

Similarly $\int_{-1}^{1} Q_{2m+1}(\mu)\, Q_{2r+1}(\mu)\, d\mu$

$$=2\sum_{n=0}^{\infty} \frac{4n+1}{(2m-2n+1)(2m+2n+2)(2r-2n+1)(2r+2n+2)}$$

$$\int_{-1}^{1} Q_{2m}(\mu)\, Q_{2r+1}(\mu)\, d\mu = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{4n+3}{(2m-2n-1)(2m+2n+2)}$$

$$\times \frac{4p+1}{(2r-2p+1)(2r+2p+2)} \int_{-1}^{1} P_{2n+1}(\mu)\, P_{2p}(\mu)\, d\mu$$

$$=0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(A)}$$

Again from
$$\int_{0}^{1} P_m(\mu)\, P_n(\mu)\, d\mu = \begin{cases} 1/2n+1 & (m=n) \\ 0 & (m-n \text{ even}) \\ \dfrac{(-)^{\mu+v}}{2^{m+n-1}(n-m)(n+m+1)} \cdot \dfrac{n!\, m!}{(v!)^2 (\mu!)^2} & \left(\begin{matrix} n=2v+1 \\ \text{or} \\ m=2\mu \end{matrix}\right) \end{cases}$$

$$\int_{0}^{1} Q_{2m}(\mu)\, Q_{2r}(\mu)\, d\mu$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(4n+3)(4p+3)}{(2m-2n-1)(2m+2n+2)(2r-2p-1)(2r+2p+2)} \times$$

$$\int_{0}^{1} P_{2n+1}(\mu)\, P_{2p+1}(\mu)\, d\mu = 0$$

Similarly, $\int_{0}^{1} Q_{2m+1}(\mu)\, Q_{2r+1}(\mu) = 0$

$$\int_0^1 Q_{2m}(\mu)\, Q_{2r+1}(\mu)\, d\mu$$

$$= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(4n+3)(4p+1)}{(2m-2n-1)(2m+2n+2)(2r-2p+1)(2r+2p+2)}$$

$$\times \int_0^1 P_{2n+1}(\mu)\, P_{2p}(\mu)\, d\mu$$

$$= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(4n+3)(4p+1)}{(2m-2n-1)(2m+2n+2)(2r-2p+1)(2r+2p+2)}$$

$$\times \frac{(-)^{n+p}}{2^{2(n+p)}(2n-2p+1)(2n+2p+2)} \frac{(2p)!\,(2n+1)!}{(n!)^2\,(p!)^2}$$

Similar result for $\int_0^1 Q_{2m+1}(\mu)\, Q_{2r}(\mu)\, d\mu$.

Lastly, since

$$\int_{-1}^1 P_n^m(\mu)\, P_r^m(\mu)\, d\mu = \begin{cases} 0 & r \neq n \\ \dfrac{2}{2n+1} \dfrac{(n+m)!}{(n-m)!} & r = n. \end{cases}$$

$$\int_{-1}^1 Q^r_{2m}(\mu)\, Q^r_{2s}(\mu)\, d\mu$$

$$= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(4n+3)(4p+3)}{(2m-2n-1)(2m+2n+2)(2s-2p-1)(2s+2p+2)}$$

$$\times \int_{-1}^1 P^r_{2n+1}(\mu)\, P^r_{2p+1}(\mu)\, d\mu$$

$$=\sum_{n=0}^{\infty}\frac{(4n+3)^2}{(2m-2n-1)\,(2m+2n+2)\,(2s-2n-1)\,(2s+2n+2)}$$

$$\times\frac{2}{4n+3}\cdot\frac{(2n+1+r)!}{(2n-r+1)!}\text{ , (r being an integer).}$$

$$=2\sum_{n=0}^{\infty}\frac{(4n+3)}{(2m-2n-1)\,(2m+2n+2)\,(2s-2n-1)\,(2s+2n+2)}$$

$$\times\frac{(2n+r+1)!}{(2n-r+1)!}$$

Similarly. $\displaystyle\int_{-1}^{1}Q^{r}_{2m+1}(\mu)\,Q^{r}_{2s+1}(\mu)\,d\mu$

$$=\sum_{n=0}^{\infty}\frac{(4n+1)^2}{(2m-2n+1)\,(2m+2n+2)\,(2s-2n+1)\,(2s+2n+2)}$$

$$\times\frac{2}{4n+1}\cdot\frac{(2n+r)!}{(2n-r)!}$$

$$=\sum_{n=0}^{\infty}\frac{(4n+1)}{(2m-2n+1)(2m+2n+2)(2s-2n+1)(2s+2n+2)}\cdot\frac{(2n+r)!}{(2n-r)!}$$

It can be proved very easily from above

$$\int_{-1}^{1}Q^{r}_{2m}(\mu)\,Q^{r}_{2s+1}(\mu)\,d\mu=0$$

$$\int_{-1}^{1}Q^{r}_{2m+1}(\mu)\,Q^{r}_{2s}(\mu)\,d\mu=0.$$

$$\int_{-1}^{1} [Q^r_{2m}(\mu)]^2 \, d\mu$$

$$= 2 \sum_{n=0}^{\infty} \frac{4n+3}{(2m-2n-1)^2 \, (2m+2n+2)^2} \cdot \frac{(2n+r+1)!}{(2n-r+1)!}$$

$$\int_{-1}^{1} [Q^r_{2m+1}(\mu)]^2 \, d\mu$$

$$= 2 \sum_{n=0}^{\infty} \frac{4n+1}{(2m-2n+1)^2 \, (2m+2n+2)^2} \cdot \frac{(2n+r)!}{(2n-r)!}$$

§. 4.

If m and n are positive integers and $m \leq n$, Adam[1] found out

$$P_m(z)\, P_n(z) = \sum_{r=0}^{\infty} \frac{A_{m-r}\, A_r\, A_{n-r}}{A_{n+m-r}} \left(\frac{2n+2m-4r+1}{2n+2m-2r+1} \right) P_{n+m-2r};$$

where $A_m = 1 \cdot 3 \cdot 5 \; \ldots\ldots \; (2m-1)/m!$

The above can be written in the form

$$P_m(z)\, P_n(z)$$

$$= \sum_{q=n-m}^{n+m} \frac{2q+1}{m+n+q+1} \cdot \frac{A_{\frac{n+m-q}{2}} \; A_{\frac{n+q-m}{2}} \; A_{\frac{m+q-n}{2}}}{A_{\frac{n+m+q}{2}}} \; P_q(z),$$

$$= \sum_{q=n-m}^{n+m} \cdot [A]_{n,\, m,\, q} \; P_q(z),$$

where $[A]_{n,\, m,\, q}$

$$= \frac{2q+1}{m+n+q+1} \cdot \frac{A_{\frac{n+m-q}{2}} \; A_{\frac{n+q-m}{2}} \; A_{\frac{m+q-n}{2}}}{A_{\frac{n+m+q}{2}}}$$

[1] Proc. Roy. Soc. XXVII.

whence. $Q_{2m}(\mu)\, Q_{2s}(\mu)\, P_{2r}(\mu)$

$$= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(4n+3)\,(4p+3)}{(2m-2n-1)\,(2m+2n+2)\,(2s-2p-1)\,(2s+2p+2)} \times P_{2n+1}(\mu)\, P_{2p+1}(\mu)\, P_{2r}(\mu)$$

Now $P_{2n+1}(\mu)\, P_{2p+1}(\mu)\, P_{2r}(\mu)$

$$= \sum_{q=n-m}^{n+m} [A]_{2n+1,\, 2p+1,\, q} \cdot P_q(\mu)\, P_{2r}(\mu)$$

$$= \sum_{q=n-m}^{n+m} [A]_{2n+1,\, 2p+1,\, q} \sum_{t=2r-q}^{2r+q} [A]_{2r,\, q,\, t}\, P_t(\mu), \qquad \text{if } r \geq n+p+1$$

This can be expanded by the scheme pointed out by S. K. Banerji.[1]

Also $\displaystyle\int_{-1}^{1} Q_{2m}(\mu)\, Q_{2s}(\mu)\, P_{2r}(\mu)\, d\mu$

$$= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(4n+3)\,(4p+3)}{(2m-2n-1)\,(2m+2n+2)\,(2s-2p-1)\,(2s+2p+2)} \times \int_{-1}^{1} P_{2n+1}(\mu)\, P_{2p+1}(\mu)\, P_{2r}(\mu)\, d\mu$$

where $\displaystyle\int_{-1}^{1} P_{2n+1}(\mu)\, P_{2p+1}(\mu)\, P_{2r}(\mu)$

$$= \frac{2}{2n+2p+2r+3} \; \frac{A_{\frac{2n+2p-2r+2}{2}}\; A_{\frac{2p+2r-2n}{2}}\; A_{\frac{2r-2n-2p-2}{2}}}{A_{\frac{2n+2p+2r+2}{2}}}$$

$$= \frac{2}{2n+2p+2r+3} \; \frac{A_{n+p-r+1}\; A_{p+r-n}\; A_{r-n-p-1}}{A_{n+p+r+1}} .$$

[1] Bull. Cal. Math. Soc. Vol. XI, No. 3, 1920, p. 180.

[2] Banerji, *loc. cit.*

Similarly the values of $\int_{-1}^{1} Q_{2m+1}(\mu)\, Q_{2s+1}(\mu)\, P_{2r+1}(\mu)\, d\mu$,

$$\int_{-1}^{1} Q_{2m+1}(\mu)\, Q_{2s+1}(\mu)\, P_{2r}(\mu), \quad \int_{-1}^{1} Q_{2m}(\mu)\, Q_{2s+1}(\mu)\, P_{2r}(\mu), \text{ etc,}$$

can be written down.

Proceeding as before the values of

$$\int_{-1}^{1} Q_m(\mu)\, Q_n(\mu)\, Q_r(\mu), \qquad [m \neq n \neq r]$$

for all possible combinations of m, n, r, according as they are even or odd, can be found out in the form of series of functions of m, n, and r.

On the steady motion of a viscous fluid due to the rotation of two spheroids about their common axis of revolution

BY

Nripendranath Sen, M.Sc.

1. In a memoir,[1] Dr. G. B. Jeffery has completely solved several cases of steady motion of a viscous fluid due to various rotating bodies including the case of two spheres rotating about their line of centres. The object of the present paper is to present the solution of a more difficult problem *viz.* the problem of the steady motion of a viscous fluid due to the rotation of two spheroids, both prolate or both oblate, about their common axis of revolution. The problem has been completely solved first for two rotating prolate spheroids with no limitation as regards their eccentricities and central distance and the solution for the case of two rotating oblate spheroids has been deduced therefrom by suitable substitutions. The success of the problem depends on a transformation theorem in Spheroidal Harmonics[2] proved in (8), (9), (10) and (11) of the present paper.

The present problem in its much simpler aspect has been attempted in a previous issue of this bulletin.[3] The results obtained there are deducible as particular cases of the general problem here studied but they are found to differ widely in many respects from the results deduced from those obtained by me. The reason for this difference is due to mistakes in approximation in that paper on its author's part, which have been clearly pointed out and explained at length in (5) and (6) of the present paper, where the correct solutions of those problems have also been given.

[1] G. B. Jeffery.—"On the steady motion of a solid of revolution in a viscous fluid"—Proc. London Math. Soc. February, 1915.

[2] Also see Bibhutibhushan Datta.—"On a transformation theorem relating to spheroidal Harmonics"—Tôhuku. Math. Jour. Vol. 15, 166-171, 1919.

[3] Bijon Dutt.—"Bul. Cal. Math. Soc." Vol. 10, 43-53, 1918-19.

TWO PROLATE SPHEROIDS ROTATING ABOUT THEIR COMMON AXIS OF REVOLUTION.

2. Let $x_1 = k_1 (1-\mu_1^2)^{\frac{1}{2}} (\lambda_1^2-1)^{\frac{1}{2}} \cos w$;

$$y_1 = k_1 (1-\mu_1^2)^{\frac{1}{2}} (\lambda_1^2-1)^{\frac{1}{2}} \sin w;$$

$$z_1 = k_1 \mu_1 \lambda_1 \qquad ;$$

$$x_2 = k_2 (1-\mu_2^2)^{\frac{1}{2}} (\lambda_2^2-1)^{\frac{1}{2}} \cos w.$$

$$y_2 = k_2 (1-\mu_2^2)^{\frac{1}{2}} (\lambda_2^2-1)^{\frac{1}{2}} \sin w.$$

$$z_2 = k_2 \mu_2 \lambda_2$$

be the two systems of co-ordinates referred to the centres of the two spheroids as origin and (λ_1, μ_1, w) (λ_2, μ_2, w) be the two systems of ovary spheroidal co-ordinates so that $\lambda_1 = \lambda_{10}$, $\lambda_2 = \lambda_{20}$ on the surfaces of the given spheroids whose semi axes are a_1, c_1 $(a_1 > c_1)$ and a_2, c_2 $(a_2 > c_2)$ respectively. Also let s=distance between their centres so that $s > a_1 + a_2$ and w_1, w_2 be their angular velocities of rotation.

If (ρ, Φ, z) be the cylindrical co-ordinates of a point, Dr. Jeffery[1] has shewn that v=velocity in the direction of ϕ,

$=f(\rho, z)$ where $v \sin \phi$ is a solution of Laplace's equation

$$\nabla^2 (v \sin \phi) = 0 \qquad \dots \quad (1)$$

and

$$v = \rho_1 w_1 = k_1 (1-\mu_1^2)^{\frac{1}{2}} (\lambda_1^2-1)^{\frac{1}{2}} w_1$$

$$\text{over } \lambda_1 = \lambda_{10} \qquad \dots \quad (2)$$

$$v = \rho_2 w_2 = k_2 (1-\mu_2^2)^{\frac{1}{2}} (\lambda_2^2-1)^{\frac{1}{2}} w_2$$

$$\text{over } \lambda_2 = \lambda_{20} \qquad \dots \quad (3)$$

The problem is, therefore, to find an expression for v satisfying the conditions (1), (2), and (3) and the further condition $v=0$ at infinity

i.e. when $\lambda = \infty$ ... (4)

[1] Jeffery—"On the steady motion of a solid of revolution in a viscous fluid" Proc. Lond. Math. Soc. Feb. 1915.

$$\text{Assume } v = \sum_{n=1}^{\infty} \{A_n P_n^1(\mu_1) Q_n^1(\lambda_1) + B_n P_n^1(\mu_2) Q_n^1(\lambda_2)\} \quad (5)$$

where A_n, B_n are arbitrary constants.

Evidently (5) satisfies (1) and (4). To determine the sets of constants A_n, B_n so as to satisfy the boundary conditions (2) and (3).

Now, it may be proved that

$$P_n^\sigma(\mu_2) Q_n^\sigma(\lambda_2) =$$

$$(-)^\sigma \frac{(n+\sigma)!}{(n-\sigma)!} \sum_{m=\sigma}^{m=\infty} (2m+1) \frac{(m-\sigma)!}{(m+\sigma)!} \omega_1(m,n) P_m^\sigma(\mu_1) P_m^\sigma(\lambda_1) \quad \dots (6)$$

when $s > a_1$ and $> a_2$

$$\text{where} \quad \omega_1(m,n) = \tfrac{1}{2} \int_{-1}^{1} Q_n\left(\frac{t_1 - p}{\rho_1}\right) P_m(p)\, dp$$

$$= (-)^r \frac{2^n n!}{(2n+1)!} \rho_1^{n+1} [D_{t_1}^n$$

$$+ \frac{\rho_1^2}{2.(2n+3)} D_{t_1}^{n+2} + \dots] Q_m(t_1)$$

where $\rho_1^2 = \dfrac{a_2^2 - c_2^2}{a_1^2 - c_1^2} = \dfrac{a_2^2 e_2^2}{a_1^2 e_1^2} = \dfrac{k_2^2}{k_1^2}$ where e_1, e_2 are eccentricities of the generating ellipses of the two spheroids and

$$t_1 = \frac{s}{k_1} \text{ and } D_{t_1}^n = \frac{d^n}{d t_1^n} \quad \dots (7)$$

$$\text{For,} \int_{-\pi}^{\pi} Q_n\left(\frac{z_2 + i x_2 \cos u + i y_2 \sin u}{k_2}\right) \cos \sigma u \, du$$

$$= \int_{-\pi}^{\pi} Q_n \{\mu_2 \lambda_2 + (\mu_2^2 - 1)^{\frac{1}{2}} (\lambda_2^2 - 1)^{\frac{1}{2}} \cos(w - u)\} \cos \sigma u \, du$$

$$= 2\pi \frac{(n-\sigma)!}{(n+\sigma)!} P_n^\sigma(\mu_2) Q_n^\sigma(\lambda_2) \cos \sigma w \quad \dots (8)^1$$

with Hobson's definition of associated Legendre's functions.

[1] Whittaker—'Mod. Analysis' 15·71, Example p. 323.

Choosing z-axes such that $z_2=s-z_1$, we have

$$\frac{(z_2+ix_2\cos u+iy_2\sin u)}{k_2}=\frac{k_1}{k_2}\cdot\frac{(s-z_1+ix_1\cos u+iy_1\sin u)}{k_1}$$

Remembering that $\dfrac{1}{\left(\dfrac{z_2+ix_2\cos u+iy_2\sin u}{k_2}\right)^{n+1}}$

$$=\rho_1^{\,n+1}\left(\frac{s}{k_1}-\frac{z_1-ix_1\cos u-iy_1\sin u}{k_1}\right)^{-n-1}$$

$$=(-)^n\frac{\rho_1^{\,n+1}}{n!}\sum_{m=0}^{\infty}(2m+1)\,P_m\left(\frac{z_1-ix_1\cos u-iy_1\sin u}{k_1}\right)\frac{d^n}{dt_1^{\,n}}Q_m(t_1)\qquad\ldots\quad(9)^1$$

defining ρ_1, t_1 as in (7), we have

$$\int^{\pi}Q_n\left(\frac{z_2+ix_2\cos u+iy_2\sin u}{k_2}\right)\cos\sigma u\,du$$

$$=\int_{-\pi}(-)^n\frac{2^n|\underline{n}}{|\underline{2n+1}}\rho_1^{\,n+1}\left\{\sum_{m=0}^{\infty}(2m+1)\,P_m\left(\frac{z_1-ix_1\cos u-iy_1\sin u}{k_1}\right)\frac{d^n}{dt_1^{\,n}}Q_m(t_1)\right.$$

$$+\frac{\rho_1^{\,2}}{2.(2n+3)}\sum_{m=0}^{\infty}(2m+1)\,P_m\left(\frac{z_1-ix_1\cos u-iy_1\sin u}{k_1}\right)\frac{d^{n+2}}{dt_1^{\,n+2}}Q_m(t_1)$$

$$\left.+\text{etc.}\right\}\cos\sigma u\,du$$

$$=\sum_{m=0}^{\infty}\omega_1(m,n)\,(2m+1)\,P_m\left(\frac{z_1-ix_1\cos u-iy_1\sin u}{k_1}\right)\cos\sigma u\,du$$

$$-2\pi\sum_{m=\sigma}^{\infty}(-)^{\sigma}\frac{|\underline{m-\sigma}}{|\underline{m+\sigma}}(2m+1)\,\omega_1(m,n)\,P_m^{\ \sigma}(\mu_1)\,P_m^{\ \sigma}(\lambda_1)\cos\sigma w\qquad\ldots\quad(10)$$

with Hobson's definition of associated Legendre Functions.

[1] Todhunter—" The functions of Laplace, Lamé and Bessel " p. 88.

$$\because \int_{-\pi}^{\pi} P_m\left(\frac{z_1 - ix_1 \cos u - iy_1 \sin u)}{k_1}\right) \cos \sigma u \, du$$

$$= \int_{-\pi}^{\pi} P_m \{\mu_1 \lambda_1 - (\mu_1{}^2 - 1)^{\frac{1}{2}} (\lambda_1{}^2 - 1)^{\frac{1}{2}} \cos (w-u)\} \cos \sigma u \, du$$

$$= 2\pi \,(-)^\sigma \frac{\lfloor m-\sigma}{\lfloor m+\sigma} P_m{}^\sigma (\mu_1) P_m{}^\sigma (\lambda_1) \cos \sigma w \qquad \ldots \quad (11)^1$$

with Hobson's definition of the associated Legendre Functions.

$\therefore$ From (8) and (10), we have,

$$P_n{}^\sigma (\mu_2) Q_n{}^\sigma (\lambda_2) = \frac{\lfloor n+\sigma}{\lfloor n-\sigma} \sum_{m=\sigma}^{\infty} (-)^\sigma \frac{\lfloor m-\sigma}{\lfloor m+\sigma} (2m+1) \, \omega_1 (m, n)$$

$$P_m{}^\sigma (\mu_1) P_m{}^\sigma (\lambda_1)$$

with either Hobson's or Ferrer's definitions of associated Legendre Functions or with both the definitions simultaneously always taking care to stick to the definition or definitions used. Thus if $P_n{}^\sigma (\mu)$ be in Ferrer's definition and $Q_n{}^\sigma (\lambda)$ or $P_n{}^\sigma (\lambda)$ be in Hobson's definition,—which is usually the case the theorem (6) is still true.

Substituting the value of $P_n{}^1 (\mu_2) Q_n{}^1 (\lambda_2)$ in (5) from (6), we have

$$v = \sum_{n=1}^{\infty} \{A_n P_n{}^1 (\mu_1) Q_n{}^1 (\lambda_1)$$

$$- B_n \sum_{m=1}^{\infty} \frac{n(n+1)(2m+1)}{m(m+1)} \omega_1 (m, n) P_m{}^1 (\mu_1) P_m{}^1 (\lambda_1)\}$$

Hence, from the surface condition (2),

$$v = k_1 (\lambda_{10}{}^2 - 1)^{\frac{1}{2}} w_1 (1 - \mu_1{}^2)^{\frac{1}{2}} = k_1 (\lambda_{10}{}^2 - 1)^{\frac{1}{2}} w_1 P_1{}^1 (\mu_1)$$

$$= \sum_{n=1}^{\infty} \{A_n P_n{}^1 (\mu_1) Q_n{}^1 (\lambda_{10})$$

$$- B_n \sum_{m=1}^{\infty} \frac{n(n+1)}{m(m+1)} (2m+1) \, \omega_1 (m, n) P_m{}^1 (\mu_1) P_m{}^1 (\lambda_{10})\}$$

[1] Whittaker—Mod. Anal. 15·7.

This must be true at every point of the surface of the spheroid $\lambda_1 = \lambda_{10}$. Equating, therefore, the co-efficients of the various zonal harmonics of μ_1,

$$k_1\, (\lambda_{10}^2-1)^{\frac{1}{2}}\, w_1 = A_1\, Q_1^1\, (\lambda_{10})$$

$$-\tfrac{3}{2} \sum_{n=1}^{\infty} n\, (n+1)\, \omega_1\, (1, n)\, P_1^1\, (\lambda_{10})\, B_n \qquad \ldots \quad (12)$$

$$0 = A_p\, Q_p^1\, (\lambda_{20})$$

$$-\frac{2p+1}{p\,(p+1)} \sum_{n=1}^{\infty} n\, (n+1)\, \omega_1\, (p, n)\, P_p^1\, (\lambda_{10})\, B_n \qquad \ldots \quad (13)$$

$$(p=2, 3, \ldots \text{ ad inf.}).$$

The corresponding set of equations giving B_1, B_2 etc. in terms of A's can be written down from symmetry from (12) and (13). Thus,

$$k_2\, (\lambda_{20}^2-1)^{\frac{1}{2}}\, w_2 = B_1\, Q_1^1\, (\lambda_{20})$$

$$-\tfrac{3}{2} \sum_{n=1}^{\infty} n\, (n+1)\, \omega_2\, (1, n)\, P_1^1\, (\lambda_{20})\, A_n \qquad \ldots \quad (14)$$

$$0 = B_p\, Q_p^1\, (\lambda_{20})$$

$$-\frac{2p+1}{p\,(p+1)} \sum_{n=1}^{\infty} n\, (n+1)\, \omega_2\, (p, n)\, P_p^1\, (\lambda_{20})\, A_n \qquad \ldots \quad (15)$$

$$(p=2, 3, \ldots \text{ ad. inf.}).$$

where $$\omega_2\, (m, n) = \tfrac{1}{2} \int_{-1}^{1} Q_n \left(\frac{t_2-\mu}{\rho_2} \right) P_m\, (\mu)\, d\mu$$

$$= (-)^n\, \frac{2^n\, |\underline{n}}{|\underline{2n+1}}\, \rho_2^{n+1}\, \left[D_{t_2}^n + \frac{\rho_2^2}{2\cdot(2n+3)}\, D_{t_2}^{n+2} \right.$$

$$\left. + \text{ etc.} \right] Q_m\, (t_2)$$

$$\rho_2 = \frac{k_1}{k_2}\,,\ t_2 = \frac{s}{k_2}\,,\ D_{t_2}^n = \frac{d^n}{d_{t_2}^n} \qquad \ldots \quad (16)$$

The two sets of equations (12), (13) and (14), (15) are sufficient to determine the two sets of constants A_1, A_2, etc and B_1, B_2, etc., as will be shewn presently.

To determine A_1, A_2, etc., substitute the values of B_1, B_2, etc. in (12) and (13) from (14) and (15). After a little simplification,

$$A_1 - \sum_{n=1} \theta_{1n} A_n = d_1 \qquad \dots \quad (17)$$

$$A_p - \sum_{n=1} \theta_{pn} A_n = dp \qquad (p=2, 3 \dots \text{ad. inf.}) \qquad \dots \quad (18)$$

where θ_{pn}

$$= \frac{n(n+1)(2p+1)}{p(p+1)} \frac{P_p^1(\lambda_{10})}{Q_p^1(\lambda_{10})} \sum_{m=1}^{\infty} (2m+1)\, \omega_1(p, m)\, \omega_2(m, n) \frac{P_m^1(\lambda_{20})}{Q_m^1(\lambda_{20})}$$

$$d_1 = k_1 w_1 \frac{P_1^1(\lambda_{10})}{Q_1^1(\lambda_{10})} + 3 k_2 w_2 \omega_1(1, 1) \frac{P_1^1(\lambda_{10})}{Q_1^1(\lambda_{10})} \frac{P_1^1(\lambda_{20})}{Q_1^1(\lambda_{20})}$$

$$d_p = \frac{2(2p+1)}{p(p+1)} k_2 w_2 \omega_1(p, 1) \frac{P_p^1(\lambda_{10})}{Q_p^1(\lambda_{10})} \frac{P_1^1(\lambda_{20})}{Q_1^1(\lambda_{20})}$$

$$(p=2, 3, \dots \text{ad. inf.}) \qquad \dots \quad (19)$$

The corresponding equations giving B_1, B_2, etc. can be written down easily from (17), (18) and (19). Thus, from symmetry,

$$B_1 - \sum_{n=1} \theta'_{1n} B_n = d'_1 \qquad \dots \quad (20)$$

$$B_p - \sum_{n=1} \theta'_{pn} B_n = d'_p \qquad (p=2, 3, \dots \text{ad inf.}) \qquad \dots \quad (21)$$

where θ'_{pn}

$$= \frac{n(n+1)(2p+1)}{p(p+1)} \frac{P_p^1(\lambda_{20})}{Q_p^1(\lambda_{20})} \sum_{m=1}^{\infty} (2m+1)\, \omega_2(p, m)\, \omega_1(m, n) \frac{P_m^1(\lambda_{10})}{Q_m^1(\lambda_{10})}$$

$$d'_1 = k_2 w_2 \frac{P_1^1(\lambda_{20})}{Q_1^1(\lambda_{20})} + 3 k_1 w_1 \omega_2(1, 1) \frac{P_1^1(\lambda_{20})}{Q_1^1(\lambda_{20})} \frac{P_1^1(\lambda_{10})}{Q_1^1(\lambda_{10})}$$

$$d'_p = \frac{2(2p+1)}{p(p+1)} k_1 w_1 \omega_2(p, 1) \frac{P_1^1(\lambda_{10})}{Q_1^1(\lambda_{10})} \frac{P_p^1(\lambda_{20})}{Q_p^1(\lambda_{20})}$$

$$(p=2, 3, \dots \text{ad. inf.}) \qquad \dots \quad (22)$$

The theory of solutions of the equations of this class has been worked out by Hill,[1] Poincare,[2] Vonkocti,[3] Teoplitz,[4] Hilbert,[5] and others. Hence, A's and B's can be completely evaluated and the problem thus becomes determinate.

3. The complete algebraic values of A's and B's thus determined are not much suitable for numerical calculations. But the constants may be calculated to any degree of approximation as follows:

$$\text{From (7), } \omega_1(m, n) = (-)^n \frac{2^n \lfloor n}{\lfloor 2n+1} \rho_1^{n+1} \Big[D_{t_1}^{n} + \frac{\rho_1^2}{2(2n+3)} D_{t_1}^{n+2} + \text{etc.} \Big] Q_m(t_1)$$

$$\text{where } \rho_1 = \frac{k_2}{k_1},\ t_1 = \frac{s}{k_1} \quad = \frac{2^{m+n} \lfloor m \lfloor n \lfloor m+n}{\lfloor 2m+1 \lfloor 2n+1} \frac{k_1^m k_2^{n+1}}{s^{m+n+1}}$$

$$\times \Big[1 + \frac{\lfloor m+n+2}{2 \lfloor m+n} \frac{1}{s^2} \left(\frac{k_1^2}{2m+3} + \frac{k_2^2}{2n+3} \right) + \frac{\lfloor m+n+4}{2\cdot 4 \lfloor m+n} \frac{1}{s^4}$$

$$\frac{k_1^4}{(2m+3)(2m+5)} + \frac{2k_1 k_2^2}{(2m+3)(2n+3)} + \frac{k_2^4}{(2n+3)(2n+5)}$$

$$+ \frac{\lfloor m+n+6}{2\cdot 4\cdot 6 \lfloor m+n} \frac{1}{s^6} \Big\{ \frac{k_1^6}{(2m+3)(2m+5)(2m+7)}$$

$$+ \frac{3k_1^4 k_2^2}{(2m+3)(2m+5)(2n+3)} + \frac{3k_1^2 k_2^4}{(2m+3)(2n+3)(2n+5)}$$

$$+ \frac{k_2^6}{(2n+3)(2n+5)(2n+7)} \Big\} + \text{etc.} \Big] \qquad \dots \quad (23)$$

(substituting the value of $Q_m(t_1)$ and simplifying).

[1] Hill—"Acta Math. 8, 1-36, 1886.

[2] Poincare—"Bul. Soc. Math. France" 14, 77-90, 1886.

[3] Vonkocti—"Rend. Circ. di Palermo" 28, 255-266, 1909.

[4] Teoplitz— Do. 28, 88-96, 1909.

[5] Hilbert—"Gött. Nachr." p. 157-227, 1906.

Thus the lowest order of $\omega_1\ (m,\ n)$ is $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^{m+n+1}$

and therefore the lowest order of θ_{pn} is $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^{p+n+4}$ putting $m=1$ in (19).

The corresponding expression for $\omega_2\ (m,\ n)$ can be readily written down from (23) by interchanging k_1 and k_2.

(A) If the spheroids are so separated that we can neglect the terms of the order $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^3$ and higher powers, we have

$$A_1=k_1\ w_1\ \frac{P_1{}^1\ (\lambda_{10})}{Q_1{}^1\ (\lambda_{10})}, \qquad A_p=0,\ (p=2,\ 3,\ 4\ldots\text{ad. inf.})$$

From symmetry $B_1=k_2\ w_2\ \dfrac{P_1{}^1\ (\lambda_{20})}{Q_1{}^1\ (\lambda_{20})}$, $B_p=0\ (p=2,\ 3,\ 4\ldots\text{ad. inf.})$.

$\therefore$ To this order of approximation, we have from (5)

$$v=k_1\ w_1\ \frac{P_1{}^1\ (\lambda_{10})}{Q_1{}^1\ (\lambda_{10})}\ P_1{}^1\ (\mu_1)\ Q_1{}^1\ (\lambda_1)$$

$$+k_2\ w_2\ \frac{P_1{}^1\ (\lambda_{20})}{Q_1{}^1\ (\lambda_{20})}\ P_1{}^1\ (\mu_2)\ Q_1{}^1\ (\lambda_2) \qquad \ldots \quad (24)$$

(B) If the terms of the order $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^5$ and higher powers are neglected, we have from (17), (18) and (19), (23)

$$A_1=k_1\ w_1\ \frac{P_1{}^1\ (\lambda_{10})}{Q_1{}^1\ (\lambda_{10})}+3k_2\ w_2\ \omega_1\ (1,\ 1)\ \frac{P_1{}^1\ (\lambda_{10})\ P_1{}^1\ (\lambda_{20})}{Q_1{}^1\ (\lambda_{10})\ Q_1{}^1\ (\lambda_{20})}$$

$$=k_1\ w_1\ \frac{P_1{}^1\ (\lambda_{10})}{Q_1{}^1\ (\lambda_{10})}+\tfrac{2}{3}\ k_1\ w_2\ \frac{P_1{}^1\ (\lambda_{10})\ P_1{}^1\ (\lambda_{20})}{Q_1{}^1\ (\lambda_{10})\ Q_1{}^1\ (\lambda_{20})}\ \frac{k_2{}^3}{s^3}$$

$$A_2=\tfrac{2}{9}\ k_2\ w_2\ \frac{P_1{}^1\ (\lambda_{20})\ P_2{}^1\ (\lambda_{10})}{Q_1{}^1\ (\lambda_{20})\ Q_2{}^1\ (\lambda_{10})}\ \frac{k_1{}^2\ k_2{}^2}{s^4}$$

$$A_p=0, \qquad (p=3,\ 4,\ldots\text{ad. inf.}).$$

The corresponding expressions for B_1, B_2 can be written down from symmetry

$$B_1 = k_2 w_2 \frac{P_1^1(\lambda_{20})}{Q_1^1(\lambda_{20})} + \tfrac{2}{3} k_2 w_1 \frac{P_1^1(\lambda_{10}) P_1^1(\lambda_{20})}{Q_1^1(\lambda_{10}) Q_1^1(\lambda_{20})} \frac{k_1^3}{s^3}$$

$$B_2 = \tfrac{2}{9} k_1 w_1 \frac{P_2^1(\lambda_{20}) P_1^1(\lambda_{10})}{Q_2^1(\lambda_{20}) Q_1^1(\lambda_{10})} \frac{k_1^2 k_2^2}{s^4}$$

$$B_p = 0, \qquad (p=3, 4, \ldots \text{ad. inf.}).$$

Hence, correct to $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^4$, we have

$$v = A_1 P_1^1(\mu_1) Q_1^1(\lambda_1) + A_2 P_2^1(\mu_1) Q_2^1(\lambda_1)$$
$$+ B_1 P_1^1(\mu_2) Q_1^1(\lambda_2) + B_2 P_2^1(\mu_2) Q_2^1(\lambda_2) \qquad \ldots \quad (25)$$

where A_1, A_2, B_1, B_2 are given above.

(C) If the terms of the order $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^7$ and higher powers are neglected, we have from (17), (18), (19) and (23)

$$A_1 = k_1 w_1 \frac{P_1^1(\lambda_{10})}{Q_1^1(\lambda_{10})}(1+\theta_{11})$$
$$+ 3k_2 w_2 \omega_1(1, 1) \frac{P_1^1(\lambda_{10}) P_1^1(\lambda_{20})}{Q_1^1(\lambda_{10}) Q_1^1(\lambda_{20})}$$

$$= k_1 w_1 \frac{P_1^1(\lambda_{10})}{Q_1^1(\lambda_{10})} \left(1 + \tfrac{4}{9} \frac{P_1^1(\lambda_{10}) P_1^1(\lambda_{20})}{Q_1^1(\lambda_{10}) Q_1^1(\lambda_{20})} \frac{k_1^3 k_2^3}{s^6}\right)$$
$$+ \tfrac{2}{3} \frac{P_1^1(\lambda_{10}) P_1^1(\lambda_{20})}{Q_1^1(\lambda_{10}) Q_1^1(\lambda_{20})} k_1 w_2 \left(1 + \tfrac{6}{5} \frac{k_1^2 + k_2^2}{s^2}\right) \frac{k_2^3}{s^3}$$

$$A_2 = d_2 = \tfrac{2}{9} k_2 w_2 \frac{k_1^2 k_2^2}{s^4} \left\{1 + \frac{10}{s^2}\left(\frac{k_1^2}{7} + \frac{k_2^2}{5}\right)\right\}$$
$$\frac{P_1^1(\lambda_{20}) P_2^1(\lambda_{10})}{Q_1^1(\lambda_{20}) Q_2^1(\lambda_{10})}$$

$$A_3 = d_3 = \tfrac{4}{45} k_2 w_2 \frac{P_3^1(\lambda_{10}) P_1^1(\lambda_{20})}{Q_3^1(\lambda_{10}) Q_1^1(\lambda_{20})} \cdot \frac{k_1^3 k_2^2}{s^5}$$

$$A_4=d_4=\tfrac{4}{105}\,k_2\,w_2\,\frac{P_4{}^1(\lambda_{10})\,P_1{}^1(\lambda_{20})}{Q_4{}^1(\lambda_{10})\,Q_1{}^1(\lambda_{20})}\cdot\frac{k_1{}^4\,k_2{}^2}{s^6}\,.$$

$A_p=0$ ($p=5$, 6...ad. inf.)

The values of B's can be written down from symmetry. Thus

$$B_1=k_2\,w_2\,\frac{P_1{}^1(\lambda_{20})}{Q_1{}^1(\lambda_{20})}\left(1+\tfrac{4}{9}\,\frac{P_1{}^1(\lambda_{10})\,P_1{}^1(\lambda_{20})}{Q_1{}^1(\lambda_{10})\,Q_1{}^1(\lambda_{20})}\,\frac{k_1{}^3\,k_2{}^3}{s^6}\right)$$

$$+\frac{2k_1{}^3\,P_1{}^1(\lambda_{10})\,P_1{}^1(\lambda_{20})\,k_2\,w_1}{3\,s^3\,Q_1{}^1(\lambda_{10})\,Q_1{}^1(\lambda_{20})}\left(1+\tfrac{6}{5}\,\frac{k_1{}^2+k_2{}^2}{s^2}\right).$$

$$B_2=\tfrac{2}{9}\,k_1\,w_1\,\frac{k_1{}^2\,k_2{}^2}{s^4}\left\{1+\frac{10}{s^2}\left(\frac{k_2{}^2}{7}+\frac{k_1{}^2}{5}\right)\right\}$$

$$\frac{P_1{}^1(\lambda_{10})\,P_2{}^1(\lambda_{20})}{Q_1{}^1(\lambda_{10})\,Q_2{}^1(\lambda_{20})}\,.$$

$$B_3=\tfrac{4}{45}\,k_1\,w_1\,\frac{P_1{}^1(\lambda_{10})\,P_3{}^1(\lambda_{20})}{Q_1{}^1(\lambda_{10})\,Q_3{}^1(\lambda_{20})}\,\frac{k_1{}^2\,k_2{}^3}{s^5}$$

$$B_4=\tfrac{4}{105}\,k_1\,w_1\,\frac{P_1{}^1(\lambda_{10})\,P_4{}^1(\lambda_{20})}{Q_1{}^1(\lambda_{10})\,Q_4{}^1(\lambda_{20})}\cdot\frac{k_1{}^2\,k_2{}^4}{s^6}$$

$B_5=B_6=\ldots\text{etc.}=0.$

Hence, correct to $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^6$,

$$v=\sum_{n=1}^{4}A_n\,P_n{}^1(\mu_1)\,Q_n{}^1(\lambda_1)+\sum_{n=1}^{4}B_n\,P_n{}^1(\mu_2)\,Q_n{}^1(\lambda_2)\quad\ldots\quad(26)$$

where A's and B's are given above.

In a similar way v can be obtained correct to any order of approximation.

4. *Expressions for v in spherical harmonics.* The solutions (24), (25), (26) etc. can be expressed in spherical harmonics by the help of the following theorems.

From (8), we have, with Hobson's definitions of associated Legendre Functions

$$P_n^\sigma(\mu_2)\, Q_n^\sigma(\lambda_2) \cos\sigma w$$

$$= \frac{1}{2\pi} \frac{|n+\sigma}{|n-\sigma} \int_{-\pi}^{\pi} Q_n\left(\frac{z_2+ix_2\cos u+iy_2\sin u}{k_2}\right)\cos\sigma u\, du$$

$$\text{Again}^1 \int^{\pi} \frac{\cos\sigma u\, d}{(z_2+ix_2\cos u+iy_2\sin u)^{n+1}}$$

$$=2\pi(-1)^\sigma \frac{|n-\sigma}{|n} \frac{P_n^\sigma(\cos\theta_2)}{r_2^{n+1}}\cos\sigma w.$$

with Hobson's definition of P_n^σ

Hence, $Q_n^\sigma(\lambda_2)\, P_n^\sigma(\mu_2)\cos\sigma w$

$$=\frac{|n+\sigma}{2\pi|n-\sigma} \cdot \frac{2^n\, |n\, |n}{|2n+1} k_2^{n+1}$$

$$\int_{-\pi}^{\pi}\left\{\frac{1}{\zeta^{n+1}} + \frac{(n+1)(n+2)k_2^2}{2\cdot(2n+3)}\frac{1}{\zeta^{n+3}}+\ldots\right\}\cos\sigma u\, du$$

writing ζ_2 for $(z_2+ix_2\cos u+iy_2\sin u)$

$$=(-)^\sigma \frac{2^n\, |n\, |n+\sigma}{|2n+1} k_2^{n+1}\left\{\frac{P_n^\sigma(\cos\theta_2)}{r_2^{n+1}}\right.$$

$$\left.+\frac{|n+2-\sigma}{|n-\sigma}\cdot\frac{k_2^2}{2\cdot(2n+3)}\frac{P_{n+2}^\sigma(\cos\theta_2)}{r_2^{n+3}}+\text{etc.}\right\}\cos\sigma w.$$

i.e. $Q_n^\sigma(\lambda_2)\, P_n^\sigma(\mu_2)$

$$=(-)^\sigma \frac{2^n\, |n\, |n+\sigma}{|2n+1} k_2^{n+1}\left\{\frac{P_n^\sigma(\cos\theta_2)}{r_2^{n+1}}\right.$$

$$\left.+\frac{|n+2-\sigma}{|n-\sigma}\cdot\frac{k_2^2}{2\cdot(2n+3)}\frac{P_{n+2}^\sigma(\cos\theta_n)}{r_2^{n+3}}+\text{etc.}\right\} \quad \ldots \quad (27)$$

[1] Whittaker-mod. analysis, 15·61. Example, p. 320.

with Hobson's definition of $Q_n^{\sigma}(\lambda_2)$ and Ferrer's definition of $P_n^{\sigma}(\mu)$ and $P_n^{\sigma}(\cos\theta)$ (extracting and cancelling the factor ι^{σ} from both the sides).

$$\text{Also } \frac{P_p^1(\lambda)}{Q_p^1(\lambda)} = \frac{\frac{d}{d\lambda}P_p(\lambda)}{\frac{d}{d\lambda}Q_p(\lambda)} = \frac{(2pc_p)^2\,(2p+1)^n}{2^{2p}\,(p+1)\,e^{2p+1}}$$

$$\times\frac{\left\{1-\frac{(p-1)(p-2)}{2(2p-1)}e^2+\frac{(p-1)(p-2)(p-3)(p-4)}{2\cdot4\cdot(2p-1)(2p-3)}e^4-\text{etc.}\right\}}{\left\{1+\frac{(p+2)(p+3)}{2(2p+3)}e^2+\frac{(p+2)(p+3)(p+4)(p+5)}{2\cdot4\cdot(2p+3)(2p+5)}e^4+\text{etc.}\right\}}$$

$$\left[\because \lambda=\frac{1}{e}\ e \text{ being eccentricity}\right] \qquad \ldots \quad (28)$$

5. From (27) and (28), $k_1\,w_1\,\frac{P_1^1(\lambda_{10})}{Q_1^1(\lambda_{10})}\,P_1^1(\mu_1)\,Q_1^1(\lambda_1)$

$$=a_1\,w_1\left[\left(1-\tfrac{6}{5}e_1^2\right)\frac{P_1^1(\cos\theta_1)}{r_1^2}+\frac{a_1^2\,e_1^2}{5}\,\frac{P_3^1(\cos\theta_1)}{r_1^4}\right]$$

neglecting e^3 and higher powers always.

Similarly, $k_2\,w_2\,\frac{P_1^1(\lambda_2)}{Q_1^1(\lambda_{20})}\,P_1^1(\mu_2)\,Q_1^1(\lambda_2)$

$$=a_2^3\,w_2\left[\left(1-\tfrac{6}{5}e_2^2\right)\frac{P_1^1(\cos\theta_2)}{r_2^2}+\frac{a_2^2\,e_2^2}{5}\,\frac{P_3^1(\cos\theta_2)}{r_2^4}\right]$$

Hence (24) gives

$$v=a_1^3\,w_1\left[\left(1-\tfrac{6}{5}e_1^2\right)\frac{P_1^2(\cos\theta_1)}{r_1^2}+\frac{a_1^2\,e_1^2}{5}\,\frac{P_3^1(\cos\theta_1)}{r_1^4}\right]$$

$$+a_2^3\,w_2\left[\left(1-\tfrac{6}{5}e_2^2\right)\frac{P_1^1(\cos\theta_2)}{r_2^2}+\frac{a_2^2\,e_2^2}{5}\,\frac{P_3^1(\cos\theta_1)}{r_2^4}\right]$$

If the equations of the spheroids be written in the form

$$r=a\,[1+\epsilon_1\, P_2\,(\cos\theta_1)]$$

$$r=a'\,[1+\epsilon_2\, P_2\,(\cos\theta_2)]$$

evidently $a=a_1\,(1-\frac{e_1^2}{3})\,;\; a'=a_2\,(1-\frac{e_2^2}{3})$

$$\epsilon_1=\frac{e_1^2}{3}\,; \qquad \epsilon_2=\frac{e_2^2}{3}\,.$$

Also, $a_1^3\,(1-\frac{6}{5}\,e_1^2)=\frac{5-\epsilon_1}{5+2\epsilon_1}\,a^3$ correct to ϵ_1.

$a_2^3\,(1-\frac{6}{5}\,e_2^2)=\frac{5-\epsilon_2}{5+2\epsilon_2}\,a'^3$ correct to ϵ_2.

$$\therefore v=a^3\,w_1\,\frac{5-\epsilon_1}{5+2\epsilon_1}\,\frac{P_1^1\,(\cos\theta_1)}{r_1^2}+\tfrac{3}{5}\,a^5\,\epsilon_1\,w_1\,\frac{P_3^1\,(\cos\theta_1)}{r_1^4}$$

$$+a'^3\,w_2\,\frac{5-\epsilon_2}{5+2\epsilon_2}\,\frac{P_1^1\,(\cos\theta_2)}{r_2^2}+\tfrac{3}{5}\,a'^5\,\epsilon_2\,w_2\,\frac{P_3^1\,(\cos\theta_2)}{r_2^4} \quad \ldots \quad (29)$$

Under the above conditions *viz.* neglecting ϵ_1^2 and higher powers and $\left(\frac{\text{linear dimension}}{\text{central distance}}\right)^3$ and higher powers, this problem has been attempted in a previous issue of this Bulletin.[1] The solution for v obtained there does not contain the 2nd and 4th terms of the right hand side of (29). This is due to the omission, through mistake, of the co-efficients of $\frac{P_3^1\,(\cos\theta_1)}{r_1^4}$ and $\frac{P_3^1\,(\cos\theta_2)}{r_2^4}$ correct to the above orders of approximation,—which is also obvious from the author's own own calculations of those co-efficients correct to higher approximation given in his results (43) and (44) from which the co-efficients of $\frac{P_3^1\,(\cos\theta_1)}{r_1^4}$ and $\frac{P_3^1\,(\cos\theta_2)}{r_2^4}$ are respectively

$$\frac{3w_1\,a^5\,\epsilon_1}{5-8\epsilon_1}-\frac{w_2\,a^3\,a'^3\,(5-\epsilon_2)}{s^3\,(5+2\epsilon_2)}\cdot\frac{\epsilon_1}{5-8\epsilon_1}$$

and

$$\frac{3\,w_2\,a'^5\,\epsilon_2}{5-8\epsilon_2}-\frac{w_1\,a'^3\,a^3}{s^3}\cdot\frac{5-\epsilon_1}{5+2\epsilon_1}\cdot\frac{\epsilon_2}{5-8\epsilon_2}$$

[1] Bul. Cal. Math. Soc. Vol. X, No. 1 ibid.

From these two the co-efficients correct to above orders of approximation are respectively $\dfrac{3\,w_1\,a^5\,\epsilon_1}{5}$ and $\dfrac{3\,w_2\,a'^5\,\epsilon_2}{5}$ which are identical with those obtained by me in (29). These cannot evidently be neglected in the expression for v correct to orders of approximation already referred to.

6. Again from (25), $A_1\,P_1{}^1\,(\mu_1)\,Q_1{}^1\,(\lambda_1)$

$$=\left\{k_1\,w_1\,\frac{P_1{}^1\,(\lambda_{10})}{Q_1{}^1\,(\lambda_{10})}+\tfrac{2}{3}\,k_1\,w_2\,\frac{k_2{}^3\,P_1{}^1\,(\lambda_{10})\,P_1{}^1\,(\lambda_{20})}{s^3\,Q_1{}^1\,(\lambda_{10})\,Q_1{}^1\,(\lambda_{20})}\right\}$$

$$P_1{}^1\,(\mu_1)\times Q_1{}^1\,(\lambda_1)$$

$$=a_1{}^3\,\left\{\left(1-\tfrac{6}{5}\,e_1{}^2\right)w_1-\frac{a_2{}^3}{s^2}\,w_2\left(1-\tfrac{6}{5}\,e_1{}^2\right)\left(1-\tfrac{6}{5}\,e_2{}^2\right)\right\}\frac{P_1{}^1\,(\cos\theta_1)}{r_1{}^2}$$

$$+\frac{a_1{}^5\,e_1{}^2}{5}\left(w_1-\frac{a_2{}^3}{s^3}\,w_2\right)\frac{P_3{}^1\,(\cos\theta_1)}{r_1{}^4}$$

neglecting $e_1{}^3$ and higher powers.

Also, proceeding similarly $A_2\,P_2{}^1\,(\mu_1)\,Q_2{}^1\,(\lambda_1)$

$$=-\frac{a_1{}^5\,a_2{}^3\,w_2}{s^4}\left(1-\tfrac{10}{7}\,e_1{}^2\right)\left(1-\tfrac{6}{5}\,e_2{}^2\right)\frac{P_2{}^1\,(\cos\theta_1)}{r_1{}^3}$$

$$-\tfrac{3}{7}\,\frac{a_1{}^7\,a_2{}^3}{s^4}\,w_2\,\frac{P_4{}^1\,(\cos\theta_1)}{r_1{}^5}\,e_1{}^2$$

Similarly writing down the expressions for $B_1\,P_1{}^1\,(\mu_2)\,Q_1{}^1\,(\lambda_2)$ and $B_2\,P_2{}^1\,(\mu_2)\,Q_2{}^1\,(\lambda_2)$ from symmetry, we have from (25),

$$v=a_1{}^3\,\left\{\left(1-\tfrac{6}{5}\,e_1{}^2\right)w_1-\frac{a_2{}^3}{s^3}\,w_2\left(1-\tfrac{6}{5}\,e_1{}^2\right)\left(1-\tfrac{6}{5}\,e_2{}^2\right)\right\}\frac{P_1{}^1\,(\cos\theta_1)}{r_1{}^2}$$

$$-\frac{a_1{}^5\,a_2{}^3}{s^4}\,w_2\left(1-\tfrac{10}{7}\,e_1{}^2\right)\left(1-\tfrac{6}{5}\,e_2{}^2\right)\frac{P_2{}^1\,(\cos\theta_1)}{r_1{}^3}$$

$$+\frac{a_1{}^5\,e_1{}^2}{5}\left(w_1-\frac{a_2{}^3}{s^3}\,w_2\right)\frac{P_3{}^1\,(\cos\theta_1)}{r_1{}^4}-\tfrac{3}{7}\,\frac{a_1{}^7\,a_2{}^3}{s^4}\,w_2$$

$$\frac{P_4{}^1\,(\cos\theta_1)}{r_1{}^5}\,e_1{}^2$$

$$+a_2^3\left\{\left(1-\tfrac{6}{5}e_2^2\right)w_2-\frac{a_1^5}{s^3}w_1\left(1-\tfrac{6}{5}e_1^2\right)\left(1-\tfrac{6}{5}e_2^2\right)\right\}\frac{P_1^1(\cos\theta_2)}{r_2^2}$$

$$-\frac{a_1^3 a_2^5}{s^4}w_1\left(1-\tfrac{10}{7}e_2^2\right)\left(1-\tfrac{6}{5}e_1^2\right)\frac{P_2^1(\cos\theta_2)}{r_2^3}$$

$$+\frac{a_2^5 e_2^2}{5}\left(w_2-\frac{a_1^3}{s^3}w_1\right)\frac{P_3^1(\cos\theta_2)}{r_2^4}-\tfrac{8}{7}\frac{a_1^3 a_2^7}{s^4}w_1$$

$$\frac{P_4^1(\cos\theta_2)}{r_2^5}e_2^2 \qquad \ldots \quad (30)$$

Then remembering that ϵ_1^2, $\epsilon_1\epsilon_2$, ϵ_2^2 and higher powers are always to be neglected, the co-efficient of $\dfrac{P_1^1(\cos\theta_1)}{r_1^2}=w_1 a^3\dfrac{5-\epsilon_1}{5+2\epsilon_1}$

$$-w_2\frac{a^3 a'^3}{s^3}\;\frac{5-\epsilon_1}{5+2\epsilon_1}\cdot\frac{5-\epsilon_2}{5+2\epsilon_2},$$

This differs from the corresponding expression of Mr. Dutt which does not contain the last term, the omission of which is also due to a mistake in approximation on his part. This omissiou has, as I shall presently shew, affected the correctness of Mr. Dutt's expression for co-efficient of $\dfrac{P_3^1(\cos\theta_1)}{r_1^4}$ as well.

The co-efficient of $\dfrac{P_2^1(\cos\theta_1)}{r_1^3}$

$$=-\frac{a_1^5 a_2^3}{s^4}w_2\left(1-\tfrac{10}{7}e_1^2\right)\left(1-\tfrac{6}{5}e_2^2\right).$$

$$=-\frac{a^5 a'^3 w_2}{s^4}\;\frac{5-\epsilon_2}{5+2\epsilon_2}\left(1+\frac{5\epsilon_1}{7-3\epsilon_1}\right)$$

The co-efficient of $\dfrac{P_4^1(\cos\theta_1)}{r_1^5}=-\tfrac{8}{7}\,\dfrac{a_1^7 a_2^3 w_2 e_1^2}{s^4}$

$$=-\tfrac{9}{7}\epsilon_1\frac{a^7 a'^3}{s^4}w_2$$

The last two results are identical with those obtained by Mr. Dutt.

But the co-efficient of $\frac{P_3^1 (\cos\theta_1)}{r_1^4} = \frac{a_1^5 e_1^2}{5}\left(w_1 - \frac{a_2^3}{s^3} w_2\right)$

$$= \frac{3a^5 w_1 \epsilon_1}{5-8\epsilon_1} - \frac{3a^5 a'^3}{s^3} w_2 \frac{\epsilon_1}{5-8\epsilon_1} \cdot \frac{5-\epsilon_2}{5+2\epsilon_2}.$$

($\because \epsilon_1^2$, $\epsilon_1 \epsilon_2$ etc. and higher powers are to be neglected).

The corresponding expression obtained by Mr. Dutt is

$$\frac{3w_2 a^5 e_1}{5-8\epsilon_1} - \frac{w_2 a^5 a'^3}{s^3} \frac{5-\epsilon_2}{5+2\epsilon_2} \cdot \frac{\epsilon_1}{5-8\epsilon_1}.$$

The reason for this difference is to his wrongly omitting the term of the order $\left(\frac{\text{linear dimension}}{\text{central distance}}\right)^3$ in the co-efficient of $\frac{P_1^1 (\cos\theta_1)}{r_1^2}$. For, according to his notation, the equation (42) giving the co-efficient A_3 is

$$w_1 a^3 \epsilon_1 = \frac{A_3}{a^3}(5-8\epsilon_1) - \frac{2A_1}{a}\epsilon_1 + \epsilon_1 \frac{a^2}{s^3} \cdot R_2$$

If A_1 be wrongly taken equal to $w_1 a^3 \frac{5-\epsilon_1}{5+2\epsilon_1}$, $R_2 = 2\frac{w_2 a'^3}{5}$. $\frac{5-\epsilon_2}{5+2\epsilon_2}$, A_3 is obtained in the form obtained by Mr. Dutt. But if on the other hand we take the value of the co-efficient correct to $\left(\frac{\text{linear dimension}}{\text{central distance}}\right)^3$ which cannot of course be neglected correct to the order of approximation already referred to in the Case (B) *i.e.* if in his equation (42),

$$A_1 = w_1 a^3 \frac{5-\epsilon_1}{5+2\epsilon_1} - w_2 \frac{a^3 a'^3}{s^3} \frac{5-\epsilon_1}{5+2\epsilon_1} \cdot \frac{5-\epsilon_2}{5+2\epsilon_2}, \text{ we obtain}$$

$$A_3 = \frac{3a^5 w\epsilon_1}{5-8\epsilon_1} - \frac{3 w_2 a^5 a'^3}{s^3} \cdot \frac{\epsilon_1}{5-8\epsilon_1} \cdot \frac{5-\epsilon_2}{5+2\epsilon_2}, \text{ which is the value}$$

obtained by me.

Thus from his own equation, we get the correct value of the co-efficient of $\frac{P_3^1(\cos\theta_1)}{r_1^4}$ by using the correct expression for the co-efficient of $\frac{P_1^1(\cos\theta_1)}{r_1^2}$.

In a similar way other expressions for r may be expressed in spherical harmonics by the help of (27) and (28).

TWO OBLATE SPHEROIDS ROTATING ABOUT THEIR COMMON AXIS OF REVOLUTION.

7. Let $x_1 = k'_1\,(1-\mu_1^2)^{\frac{1}{2}}\,(\lambda_1^2+1)^{\frac{1}{2}}\cos w$.

$y_1 = k'_1\,(1-\mu_1^2)^{\frac{1}{2}}\,(\lambda_1^2+1)^{\frac{1}{2}}\sin w$;

$z_1 = k'_1\,\mu_1\,\lambda_1$;

$$x_2 = k'_2\,(1-\mu_2^2)^{\frac{1}{2}}\,(\lambda_2^2+1)^{\frac{1}{2}}\cos w$$

$$y_2 = k'_2\,(1-\mu_2^2)^{\frac{1}{2}}\,(\lambda_2^2+1)^{\frac{1}{2}}\sin w$$

$$z_2 = k'_2\,\mu_2\,\lambda_2$$

be the two systems of co-ordinates referred to the centres of the two spheroids as origin and (λ_1, μ_1, w), (λ_2, μ_2, w) be the two systems of planetary spheroidal co-ordinates. It is easy to see that if $\frac{k'_1}{i}$ be written for k_1 and $i\lambda$ for λ in the prolate spheroidal co-ordinates, the corresponding expressions for oblate spheroidal co-ordinates are readily deducible. Hence, angular velocities etc. remaining same, the expressions for r in the present case are at once obtained by writing $\frac{k'}{i}$ and $i\lambda$ for k and λ respectively in the corresponding expressions for r in prolate spheroidal case discussed before.

4

On Caustics formed by Diffraction

BY

Panchanan Das, M.Sc.

From the examination of various diffraction plates of the Fresnel class, Prof. C. V. Raman[1] and Dr. S. K. Mitra[2] came to the conclusion that these patterns exhibit a marked concentration of luminosity along curves agreeing generally in position and form with the evolute of the shadow, that should be formed according to geometrical optics, of the diffracting boundary. They also observed a series of fringes running parallel to the curves of maximum luminosity and found in fact that these curves presented a marked similarity with the caustics formed by reflection and refraction. The new class of caustics here arising may be referred to as deffraction caustics. The object of the present paper is to discuss the theory of the formation of these caustics and to place the same on an exact mathematical basis. In order to illustrate the subject, a photograph[3] of the diffraction-pattern of an one-anna coin, which has an undulating edge, was also taken, which showed these caustics beautifully. By assuming the equation of the boundary of the coin aforesaid, to be of the form

$$r=a\ (1+\epsilon \cos n\theta),$$

a calculation of the distance between consecutive fringes was carried out and it was found to agree fairly with that actually measured from the plates.

The analysis is based on a paper by Rubinowicz,[4] the substance of which may, for convenience, be reproduced here.

All diffraction problems lead to an equation of the form :—

$$\nabla^2 u+k^2 u=0.$$

[1] Phys. Rev. 13, 1919.

[2] Phil. Mag. July, 1919.

[3] This was first done by Dr. S. K. Mitra, loc. cit.

[4] Ann. d. Physik. Bd. 53, pp. 257-1917.

Let $u\,(x, y, z)$ be a finite, differentiable solution of the above equation. Then the surface-integral,

$$\frac{1}{4\pi}\int_{(G)}\int\left\{\bar{u}\,\frac{\partial}{\partial n}\left(\frac{e^{\iota\kappa r}}{r}\right)-\frac{e^{\iota\kappa r}}{r}\cdot\frac{\partial\bar{u}}{\partial n}\right\}df \qquad (1)$$

over any region vanishes or equals u. according as the pole of r is without or within the region G, where $\bar{u}$ and $\frac{\partial\bar{u}}{\partial n}$ are values on the surface of G.

Kirch-hoff assumes that on the shaded side of the screen S, the light-disturbance is

$$u=\frac{1}{4\pi}\int_{(F)}\int\left\{\bar{u}\,\frac{\partial}{\partial n}\left(\frac{e^{\iota\kappa r}}{r}\right)-\frac{e^{\iota\kappa r}}{r}\cdot\frac{\partial\bar{u}}{\partial n}\right\}df \qquad (2)$$

where F is the diffracting aperture.

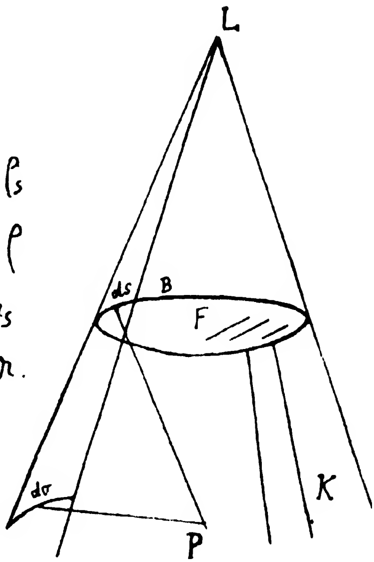
If the source of light L, (fig. 1) be a point-source then $u=\frac{e^{\iota\kappa\rho}}{\rho}$ where ρ is the distance of a point on F from L. Let K be the surface of the shadow-cone due to F bounded by the line B.

$L\,ds = \rho_s$

$L\,d\sigma = \rho$

$P\,ds = r_s$

$P\,d\sigma = r$.



Then the integral (1) extending over F and K. becomes the discontinuous function, u_E given by

Fig

$$u_E=\frac{1}{4\pi}\int_{(F+K)}\int\left\{\frac{e^{\iota\kappa\rho}}{\rho}\cdot\frac{\partial}{\partial n}\left(\frac{e^{\iota\kappa r}}{r}\right)-\frac{\partial}{\partial n}\left(\frac{e^{\iota\kappa\rho}}{\rho}\right)\cdot\frac{e^{\iota\kappa r}}{r}\right\}df \qquad \ldots \quad (3)$$

which equals $\dfrac{e^{\iota\kappa\rho}}{\rho}$ in the region (directly illuminated) bounded by F and K, and vanishes outside.

So, u_E ("incident" u) is the disturbance of geometrical optics, which neglects diffraction.

Thus, on the surface K, $\dfrac{\partial}{\partial n}\left(\dfrac{e^{\iota\kappa\rho}}{\rho}\right)=0$, and

$$\frac{\partial}{\partial n}\left(\frac{e^{\iota\kappa r}}{r}\right)=\frac{\partial}{\partial r}\left(\frac{e^{\iota\kappa r}}{r}\right)\cos(n, r)$$

$$=\left(\frac{\iota\kappa}{r}-\frac{1}{r^2}\right)e^{\iota\kappa r}\cos(n, r).$$

So from (2) and (3) we get for Kirch-hoffs' diffraction integral, the expression

$$u=u_E-\frac{1}{4\pi}\int_K\!\!\int\frac{e^{\iota\kappa(r+\rho)}}{\rho}\cdot\left(\frac{\iota\kappa}{r}-\frac{1}{r^2}\right)\cos(n, r)\,df \quad \ldots . \ (4)$$

This is transformed into a line integral round the edge E. Call it u_B="diffracted" u. For orthogonal surface coordinates, let us take ρ, and the section line σ of K with spheres ρ=const. If a linear element ds of B is distant ρ_s from L, then (fig. 1),

$$d\sigma=\frac{\rho}{\rho_s}\sin(\rho_s, ds)\,ds.$$

$$\therefore\ df=d\rho d\sigma=\frac{\rho d\rho}{\rho_s}\sin(\rho_s, ds)\,ds.$$

If r_s be the distance of any point P from ds, we have

$$r^2=r_s^2+(\rho-\rho_s)^2+2r_s(\rho-\rho_s)\cos(r_s, \rho_s).$$

also, $\cos(n, r) = \frac{r_s}{r} \cos(n, n_s)$. Thus,

$$u_B = -\frac{1}{4\pi}\int_B ds. \sin(\rho_s, ds)\cos(n, r_s).$$

$$\frac{r_s}{\rho_s}\int_{\rho_s}^{\infty} d\rho\, e^{\iota\kappa(\rho+r)}\left(\frac{\iota\kappa}{r^2} - \frac{1}{r^3}\right).$$

$$\text{Now}\int_{\rho_s}^{\infty} e^{\iota\kappa(\rho+r)}\frac{\iota\kappa}{r^2}\,d\rho = \int_{\rho_s}^{\infty}\frac{d}{d\rho}\cdot\left\{e^{\iota\kappa(\rho+r)}\right\}\cdot\frac{d\rho}{\left(1+\frac{dr}{d\rho}\right)r^2}.$$

$$= \int_{\rho_s}^{\infty}\frac{d}{d\rho}\cdot\left\{e^{\iota\kappa(\rho+r)}\right\}\cdot\frac{d\rho}{\{r+\rho-\rho_s+r_s\cos(r_s,\rho_s)\}\cdot r}$$

$$\text{and } \frac{d}{d\rho}\cdot\frac{1}{\{r+\rho-\rho_s+r_s\cos(r_s,\rho_s)\}r} = -\frac{1}{r^3}.$$

$$\text{Thus,}\int_{\rho_s}^{\infty} e^{\iota\kappa(\rho+r)}\left(\frac{\iota\kappa}{r^2} - \frac{1}{r^3}\right)d\rho$$

$$= \int_{\rho_s}^{\infty}\frac{d}{d\rho}\left\{e^{\iota\kappa(\rho+r)}\frac{1}{[r+\rho-\rho_s+r_s\cos(r_s,\rho_s)]r}\right\}d\rho.$$

$$= -\frac{e^{\iota\kappa(\rho_s+r_s)}}{r_s^2[1+\cos(r_s,\rho_s)]}.$$

Thus, $u = u_E + u_B$

$$= u_E + \frac{1}{4\pi}\int_B \frac{e^{\iota\kappa\rho_s}}{\rho_s}\cdot\frac{e^{\iota\kappa r_s}}{r_s}\cdot\frac{\cos(n, r_s)\sin(\rho_s, ds)}{\{1+\cos(r_s,\rho_s)\}} \quad \dots \quad (5)$$

The diffraction wave due to the element ds is then

$$du_B = \frac{1}{4\pi}\cdot\frac{e^{\iota\kappa\rho_s}}{\rho_s}\cdot\frac{e^{\iota\kappa r_s}}{r_s}\cdot\frac{\cos(n, r_s)}{1+\cos(r_s,\rho_s)}\sin(\rho_s, ds)\,ds \quad \dots \quad (6)$$

We shall suppose that both the source and the screen are at large distances compared with the maximum diameter of the diffracting aperture or obstacle. In order to evaluate the integral of (6) for the most general boundary we shall have to effect some reductions and simplifications.

Let the equation of the boundary referred to the tangent and normal as axes, at any point O, be

$$y = bx^2 + cx^3$$

If the screen be held parallel to the aperture at a distance h, the coordinates of any point P on the line of intersection of the screen and normal plane to the boundary curve through O, are $(0, R, h)$ say.

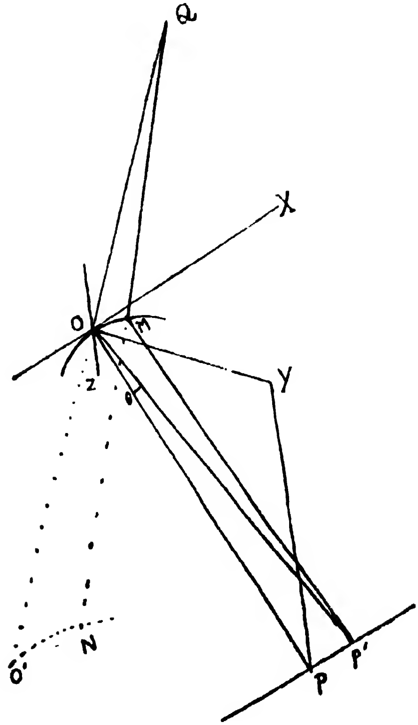


Fig 2.

Through P draw PP′ in the plane of the screen parallel to the tangent OX. Let the angle POP′$=\theta$, where θ is small. Then we have

$$OP^2 = R^2 + h^2 = l^2 \text{ and } OP = l, \text{ say.}$$

Then OP′$=l \sec\theta$, and PP′$=l\tan\theta = l\theta$, approximately.

The coordinates of M are $(x, y, 0)$; those of P are $(l\theta, R, h)$. We also have

$$MP'^2 = (x - l\theta)^2 + (R - y)^2 + h^2$$

$$= (x - l\theta)^2 + (R - bx^2 - cx^3 \ldots)^2 + h^2$$

$$= l^2 + (x - l\theta)^2 - 2R\,(bx^2 + cx^3 \ldots)$$

$$\therefore MP' = l\left\{1 + \frac{(x - l\theta)^2 - 2R\,(bx^2 + cx^3)}{2l^2}\right\} \text{ approximately.}$$

Let the source of light be Q, (x_0, y_0, z_0).

Then $OQ^2=(x_0^2+y_0^2+z_0^2)=L^2$, say.

And $MQ^2=(x_0-x)^2+(y_0-y)^2+z_0^2$

$=L^2-2xx_0-2y_0(bx^2+cx^3)+x^2$, neglecting higher powers of x than the cube.

Hence $MQ=L\left\{1+\dfrac{x^2-2y_0\,(bx^2+cx^3)-2xx_0}{2L^2}\right\}$.

Therefore, $MQ+MP'$

$$=l+L+\frac{1}{2l}\{x^2-2l\theta x+l^2\theta^2-2R\,(bx^2+cx^3)\}$$

$$+\frac{1}{2L}\{x^2-2xx_0-2y_0\,(bx^2+cx^3)\}.$$

With an object in view which will be seen later on, we equate the coefficients of x and x^2 to zero; thus:—

$$\theta=-\frac{x_0}{L} \qquad \dots \quad (7)$$

and

$$\frac{1-2Rb}{l}+\frac{1-2y_0b}{L}=0 \qquad \dots \quad (8)$$

Thus the point P′ $(l\theta, R, h)$ is determined uniquely with reference to the origin O by the equations (7) and (8). If we now regard O as a variable point on the boundary the corresponding point P′ describes a curve locus. We shall first show that this curve locus is approximately the evolute of the geometrical shadow of the boundary.

Let the point N (x', y', z') be the point on the geometrical shadow of the boundary corresponding to the point M $(x, y, 0)$ and let O′ correspond to the origin O, on the boundary; then we have

$$\frac{x-x'}{x-x_0}=\frac{y-y'}{y-y_0}=\frac{z'}{z_0}=-\frac{h}{z_0}.$$

Eliminating x, y with the help of the equation

$y=bx^2+cx^3+\dots$, we get

$$y'z_0+hy_0=b\,\frac{(x'z_0+hx_0)^2}{z_0+h}+c\,\frac{(x'z_0+hx_0)^3}{(z_0+h)^2}+\dots$$

Substituting $x'=\zeta-\frac{hx_0}{z_0}$, and $y'=\eta-\frac{hy_0}{z_0}$, we get

$$\eta=\frac{bz_0}{z_0+h}\,\xi^2+\frac{cz_0{}^2}{(z_0+h)^2}\,\zeta^3+\quad\ldots\qquad(9)$$

This is the equation of the geometrical shadow of the boundary referred to parallel axes on the screen, of which the origin is $\left(-\frac{hx_0}{z_0},-\frac{hy_0}{z_0},h\right)$ with reference to original axes. The form of (9) suggests at once, that the ξ—and η—axes are the tangent and normal to the geometrical shadow of the boundary at O′.

Let the coordinates of the centre of curvature at O′ referred to ζ—and η—axes be $(0, R_1)$, where R_1=radius of curvature at O′. Then from (9), we easily see that

$$R_1=\frac{1}{2b}\left(1+\frac{h}{z_0}\right).$$

Hence referred to the original axes, the coordinates of this centre of curvature are

$$\left(-\frac{hx_0}{z_0},\ \frac{1}{2b}+\frac{h}{2bz_0}-\frac{hy_0}{z_0},\ h\right).$$

Now, if we regard x_0, y_0 as small compared with z_0, so that $L=z_0$, and $l=h$, approximately these coordinates might be written :—

$$\left(-\frac{lx_0}{L},\ \frac{1}{2b}+\frac{l}{L}\cdot\frac{L}{2b}-\frac{l}{L}\cdot y_0,\ h\right).$$

Calculating the values of θ and R from (7) and (8) respectively, we see that the point P′ given by $(l\theta, R, h)$ and the centre of curvature at O′ are identical. Hence, the locus described by the point P′ as the origin is shifted, is the evolute of the geometrical shadow of the boundary.

We now proceed to show that the locus of P′ traces also the general outline of the region of maximum intensity in the diffraction pattern.

The light-disturbance du from an element ds of the boundary is given by (6). Let us discuss in detail and calculate the value of du for the elements of the boundary near O at the point P′.

If we expand du in terms of the element ds or rather x, and retain only a few terms of the expansion, we shall find that the change in the periodic factor $e^{\iota\kappa(\rho_1+r_1)}$ will be extremely rapid, while the

changes in the other factors are negligible, as x varies slowly. Thus we shall look upon $e^{\iota\kappa(\rho_1+r_1)}$ as the only variable factor and the rest as constant.

Now let $\rho_1+r_1=a_0+a_1x+a_2x^2+a_3x^3+\ldots$

Then since $k=\frac{2\pi}{\lambda}$, where λ is the wave-length we have.

$$k(\rho_1+r_1)=2\pi\left(\frac{a_0}{\lambda}+a_1\cdot\frac{x}{\lambda}+a_2\cdot\frac{x^2}{\lambda}+a_3\cdot\frac{x^3}{\lambda}+\ldots\right).$$

If x or x^2 be of the same order of magnitude as λ, then the above expression changes extremely rapidly with x, and thus $e^{\iota\kappa(\rho_1+r_1)}$ being an oscillating function, will, on being integrated with reference to x, have in general an inappreciable value. But if $a_1=a_2=0$, then $\frac{x^3}{\lambda}$ and the higher powers become negligible, so that $e^{\iota\kappa(\rho_1+r_1)}$ will have a constant value correct to the order of $\frac{x^3}{\lambda}$, and the integral will have an appreciable value. This fact has an important bearing on the formation of diffraction patterns.

If P′, instead of being uniquely determined with reference to the origin by (7) and (8), were any point chosen at random near P′ as determined by (7) and (8) then the constants a_1, a_2 would not in general be zero, and hence du would not have an appreciable value at P′. But if P′ is determined by (7) and (8), the value of du is appreciable there and is in fact a maximum. Hence the locus traced by P′ as the origin is shifted will be one of maximum illumination in the diffraction pattern. But we have seen above that the locus of P′ is the evolute of the geometrical shadow of the boundary approximately. Hence the diffraction pattern exhibits a strong illumination along the evolute of the shadow of the boundary of the diffracting aperture or obstacle.

We next give a theory of the diffraction caustics that are formed close to the prominent lines of the pattern.

Instead of determining the point P′ uniquely with reference to the origin, let us now abolish the condition (7) but retain (8), so that the position of the point P′ is a function of the small angle θ occurring in fig. 1. Then we can write du thus:—

$$du=A\,e^{\iota\kappa(\rho_1+r_1)}\,ds$$

$$=A\, e^{\iota\kappa\,(MQ+MP')}\, ds$$

$$=A\, e^{\iota\kappa\left\{l+L+\frac{l\theta^2}{2}-x\left(\theta+\frac{x_0}{L}\right)-cx^3\left(\frac{R}{l}+\frac{y_0}{L}\right)\right\}}\, dx$$

$$=A'\, e^{\alpha x+\beta x^3}\, dx \qquad \ldots \quad (10)$$

In order to integrate this we assume, as is generally done, that the influences of the elements of the boundary at an appreciable distance from 0 annul each other, and integrate the expression between $\pm\infty$. Thus,

$$u=A\int_{-\infty}^{\infty} e^{\alpha x+\beta x^3}\, dx \qquad \ldots \quad (11)$$

This is the well known Airy integral, and its properties have been discussed by Mascart.[1] He shows that as α is varied a series of brilliant fringes appear, separated by dark intervals. Obviously α is a linear function of θ and varying θ means studying the light-disturbance in the neighbourhood of the main diffraction pattern.

An examination of the boundary of the shadow of the coin shows that the part of the waving contour between two successive points of inflexion may ba taken as a small arc of an ellipse. The part of the diffraction pattern corresponding to this element is easily seen to have a cusp just like the evolute of an ellipse.

In order to have a rough estimate of the spacing of the caustics we may assume that the equation of the boundary of the coin is of the form :—

$$r=a\,(1+\epsilon\cos n\theta)$$

where ϵ is small and $n=12$.

To determine ϵ, the maximum and minimum diameters of the coin were measured by means of callipers and the values of a. and ϵ were calculated from these values. Thus,

$$2a\,(1+\epsilon)=20{\cdot}6\ mm.$$

$$2a\,(1-\epsilon)=19{\cdot}5\ mm.$$

whence $a=10{\cdot}02\ mm.$ and $\epsilon={\cdot}027$ approximately.

[1] Traite d'Optique. Tome I, pp. 393-4.

Now in order to evaluate the integral (11) for the above boundary we have first to find the equation of the curve with reference to the tangent and normal at any point on the boundary as axes.

Now we may regard the curve as a superposition of the displacements y_1 and y_2, where y_1 as a function of x represents the equation of a small arc of a circle with reference to axes parallel to the tangent and normal at any point and y_2 as a function of x represents the usual harmonic curve with proper dimensions. Thus let $y_1+f=\frac{1}{2a}(x+g)^2$, and $y_2=-a\epsilon\cos n\left(\frac{x}{a}+\theta_0\right)$.

$$\text{Then } y=y_1+y_2=-f+\frac{1}{2a}(x^2+2gx+g^2)$$

$$-a\epsilon\left\{\cos n\theta_0\left(1-\frac{n^2x^2}{2a^2}\right)-\sin n\theta_0\cdot\left(\frac{nx}{a}-\frac{n^3x^3}{6a^3}\right)\right\}.$$

$$=-f+g^2-a\epsilon\cos n\theta_0+\left(\frac{g}{a}+n\epsilon\sin n\theta_0\right)x+\frac{1}{2a}(1+n^2\epsilon\cos n\theta_0)x^2$$

$$-\frac{n^3\epsilon\sin n\theta_0}{6a^2}x^3$$

Now let the values of f and g be so adjusted that the absolute term and the term in x vanish.

Thus $y=bx^2+cx^3$,

where $$b=\frac{1}{2a}(1+\epsilon n^2\cdot\cos n\theta_0)$$

and $$c=-\frac{\epsilon n^3\sin n\theta_0}{6a^2}.$$

The integral, we have to study, is

$$\int e^{-\iota\kappa\{(\theta+\frac{x_0}{L})x+c\left(\frac{R}{l}+\frac{y_0}{L}\right)x^3\}}dx,$$

or taking the real part only, the expression under the integral sign, is

$$\cos\left\{k\left(\theta+\frac{x_0}{L}\right)x+kc\left(\frac{R}{l}+\frac{y_0}{L}\right)x^3\right\}.$$

Put
$$\frac{\pi}{2}u^3=kc\left(\frac{R}{l}+\frac{y_0}{L}\right)x^3$$

and
$$\frac{\pi}{2}zu=k\left(\theta+\frac{x_0}{L}\right)x.$$

Then the above expression becomes $\cos\frac{\pi}{2}(u^3+zu)$, which form has been exhaustively studied in Mascart's book. It follows that

$$z=\frac{\theta+\frac{x_0}{L}}{\sqrt[3]{\frac{\pi^2}{4k^2}\left(\frac{R}{l}+\frac{y_0}{L}\right)}}.$$

We remember that $k=\frac{2\pi}{\lambda}$ and it follows from (8) that

$$\frac{R}{l}+\frac{y_0}{L}=\frac{1}{2b}\left(\frac{1}{l}+\frac{1}{L}\right).$$

Thus

$$z=\frac{\theta+\frac{x_0}{L}}{\sqrt[3]{\frac{c\lambda^2}{32b}\left(\frac{1}{l}+\frac{1}{L}\right)}}.$$

If the first two maxima correspond to the values θ_1, θ_2 and z_1, z_2 respectively, we have

$$z_1-z_2=\frac{\theta_1-\theta_2}{\sqrt[3]{\frac{c\lambda^2}{32b}\left(\frac{1}{l}+\frac{1}{L}\right)}}.$$

If d is the linear distance between these two maxima, then $\theta_1-\theta_2=\frac{d}{l}$, approximately.

The values of z_1 and z_2 (Mascart) corresponding to these values are 5·14 and 3·47 respectively. From an examination of the pattern it

was estimated that the point on the boundary for which the value d was actually measured corresponded to the value $n\theta_0=45^\circ$ approximately. If we now measure everything in millimetres, then

$$\lambda \text{ for yellow light} = \cdot 00058 mm.$$

$$l=1330,\ L=2050,\ b=\cdot 3,\ c=\cdot 36.$$

Substitution of these in the formula readily gives

$$d=\cdot 56\ mm. \text{ approx.}$$

An actual measurement of d was carried out by means of a travelling microscope and the mean value was about $\cdot 6 mm.$, thus giving a fairly good agreement between calculated and observed values

My best thanks are due to Professor C. V. Raman for his continual help and active interest in this paper.

The Generalized Angle Concept

BY

Dr. Philip Franklin.

In a paper entitled "On the Angle-Concept in n-dimensional Geometry"[1] S. Ganguly suggested the problem of studying the inclinations of two k-spaces in an n-space ($k<n$), and gave a complete discussion for the case $k=2$. In this paper we shall extend his results to the general case, and incidentally develop some relations concerning the the volumes in the k-spaces.

We shall obtain our results by the use of tensors, and shall accordingly recall the necessary definitions and elementary properties. In a Euclidean n-space, the vectors drawn from a fixed point form a set linearly dependent on n independent ones. We start with one such set $(e_1 \ldots e_n)$ and call them the unit vectors, although they need not be of the same length or at right angles to one another when interpreted through a Cartesian system in the ordinay way. A second set of unit vectors $(e_1 \ldots e_n)$ will be related to the first set by equations of the form :

$$(1) \qquad = \sum_k A_i^k e_k \text{ or } A_i^k e_k$$

a. we shall in future omit the summation sign, it being understood for all indices which appear twice. To obtain the new components of a vector x^i we express it as a linear combination of the e's :

$$(2) \qquad x^i e_i$$

and apply (1). This gives :

$$(3) \qquad x^i = A_k^i x^k \text{ or } x^k = \breve{A}_i^k x^i$$

where the matrix of the $\breve{A}$'s is the inverse of that of the A's. The transformation (3) is said to be *contragredient* with that of (1), and the

[1] Bulletin of the Calcutta Mathematical Society, Vol. IX, 1918, p. 11.

x^i's are accordingly called *contravariant* variables. The e_i's, or any variables y_i transformed by the equations :

$$(4) \qquad y_i = \breve{A}_i^k \bar{y}_k \quad \text{or} \quad \bar{y}_i = A_i^k y_k$$

transformed *cogrediently* with (1) are called *covariant* variables.

If a set of functions $p_{k\,r}^{i\,j}$, constants in any one coordinate system, and defined for all coordinate systems is given, which are such that the expression

$$(5) \qquad p_{k\,r}^{i\,j} x_i y_j z^k u^r$$

is the same in all coordinate systems, where x and y are any covariant variables, and z and u are any contravariant variables, the set of p's are the *components* of a *tensor*, *contravariant* in i and j (in general in the superscripts) and *covariant* in k and r (in general in the subscripts). (5) refers to a tensor of the fourth order; the generalization to n-indices is obvious. Since (5) is an invariant, if it has a geometrical significance in one system, in terms of that given to the variables (*e.g.*, we may regard the contravariant ones as components of vectors, the covariant ones will be interpreted presently) it will maintain its significance in all systems.

From our definition of contravariance, it follows that

$$(6) \qquad x^i y_i$$

is an invariant, and hence that the (contravariant) components of a vector are the components of a tensor of the first order. As a tensor of the second order, consider the length of a vector. As we are dealing with oblique coordinates in Euclidean space, its square will be a quadratic form : (which may be taken as a symmetric form) :

$$(7) \qquad g_{ij} x^i x^j.$$

This will give rise to a bilinear form :

$$(8) \qquad g_{ij} x^i y^j$$

the "scalar product" of the vectors x and y. Thus the g_{ij} are the components of a tensor, whose significance in terms of the two vectors used is the product of their lengths by the cosine of the angle between them. If we form

$$(9) \qquad x_i = g_{ij} x^j \quad \text{and} \quad \bar{x}_i = \bar{g}_{ij} \bar{x}^j$$

in view of (8) we see that the x_i are covariant variables, the covariant components of the vector x^i : by solving (9), we obtain :

(10) $x^i = g^{ij} x_j$, ($\|g^{ij}\|$ the inverse of $\|g_{ij}\|$)

enabling us to calculate contravariant components from covariant ones.

There are several methods of forming new tensors from given tensors. Thus the sum of two tensors of the same order with corresponding indices is a tensor. (*e.g.*, $p^{ij}+q^{ij}=t^{ij}$). Likewise the product of any two tensors is a tensor (*e.g.*, $p^{ij}q_r=t_r^{ij}$). These facts are easily proved by noting that the expression for t to be shown an invariant is the sum or product of the invariants for p and q. Further the results obtained from a given tensor by equating a covariant and a contravariant index and summing for this index (contraction) are the components of a tensor, (*e.g.*, $p_r^{ijr}=t^{ij}$). For $t^{ij}x_iy_j=p_r^{ijr}x_iy_j$ is invariant since both $p_r^{ijr}x_iy_jz_ru^r$ and z_ru^r are invariant. We shall use these methods to build up, from g_{ij} and vectors an invariant which bears the same relation to k-space that the scalar product does to the vector, and shall obtain our results from this invariant.

Let a k-space be determined by the k independent vectors $a^i, b^i \ldots q^i$. The product of these will be a tensor, and by permuting the indices and adding or subtracting the results, we find that :

$$(11) \quad K^{ij\ldots s} = \begin{vmatrix} a^i & a^j & \ldots & a^s \\ \vdots & \vdots & & \vdots \\ q^i & q^j & \ldots & q^s \end{vmatrix}$$

is a tensor. Since

$$(12) \quad K^{ij\ldots s}x_iy_j\ldots u_s = K^{ij\ldots s}g_{im}g_{jn}\ldots g_{st}x^my^n\ldots u^t$$

is invariant, we may obtain a new tensor :

$$(13) \quad K_{mn\ldots t} = K^{ij\ldots s}g_{im}g_{jn}\ldots g_{st}$$

and since from (11) $K^{ij\ldots s}$ is zero if two of the indices are equal, and changes sign when two are interchanged (*i.e.*, is skew-symmetric), we may write in place of (13) :

$$(14) \quad k!K_{mn\ldots t} = \begin{vmatrix} a^i & a^j & \ldots & a^s \\ \vdots & \vdots & & \vdots \\ q^i & q^j & & q^s \end{vmatrix} \begin{vmatrix} g_{im} & g_{jm} & \ldots & g_{sm} \\ \vdots & \vdots & & \vdots \\ g_{it} & g_{jt} & \ldots & g_{st} \end{vmatrix} .$$

By multiplying the tensors given in (11) and (14), and contracting with respect to all the indices, we obtain the invariant :

$$(15)\quad k!\mathrm{V}^2_{a-q}=\begin{vmatrix} a^m & a^n & \dots & a^t \\ \vdots & \vdots & & \vdots \\ q^m & q^n & \dots & q^t \end{vmatrix}\begin{vmatrix} a^i & a^j & \dots & a^s \\ \vdots & \vdots & & \vdots \\ q^i & q^j & \dots & q^s \end{vmatrix}\begin{vmatrix} g_{im} & g_{jm} & \dots & g_{sm} \\ \vdots & \vdots & & \vdots \\ g_{it} & g_{jt} & \dots & g_{st} \end{vmatrix}$$

V_{a-q} *is the volume of the k-dimensional parallelopiped formed from the vectors* $a^i \dots q^i$. For if we use a Cartesian coordinate system (for which $g_{ij}=\delta_{ij}=1$ or 0 according as $i=$ or $\neq j$) whose first k axes are in the k-space determined by $a^i \dots q^i$, (15) reduces to the single term :

$$(16)\qquad k!\begin{vmatrix} a^1 & a^2 & \dots & a^k \\ \vdots & \vdots & & \vdots \\ q^1 & q^2 & \dots & q^k \end{vmatrix}^2\begin{vmatrix} 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{vmatrix}$$

after the terms resulting from the different permutations are combined, and the first determinant is the expression for the volume of the k-parallelopiped in Cartesian coordinates, $k+1$ of whose vertices are the origin, and the extremities of the vectors $a^i \dots q^i$ drawn from the origin.[1]

For two k-spaces, one given by $a^i, b^i \dots q^i$ and the other by $\mathrm{A}^i, \mathrm{B}^i \dots \mathrm{Q}^i$ we may form the expression (11) for the first, and (14) for the second, and by multiplication and contraction form the expression

$$(17)\quad k!\mathrm{S}_{a,\mathrm{A}}=\begin{vmatrix} a^m & a^n & \dots & a^t \\ \vdots & \vdots & & \vdots \\ q^m & q^n & \dots & q^t \end{vmatrix}\begin{vmatrix} \mathrm{A}^i & \mathrm{A}^j & \dots & \mathrm{A}^s \\ \vdots & \vdots & & \vdots \\ \mathrm{Q}^i & \mathrm{Q}^j & \dots & \mathrm{Q}^s \end{vmatrix}\begin{vmatrix} g_{im} & g_{jm} & \dots & g_{sm} \\ \vdots & \vdots & & \vdots \\ g_{it} & g_{jt} & \dots & g_{st} \end{vmatrix}$$

analagous to (15), which is the generalization of the scalar product. If we write

$$(18)\qquad \cos\Omega=\frac{\mathrm{S}_{a,\mathrm{A}}}{\mathrm{V}_{a-q}\mathrm{V}_{\mathrm{A}-\mathrm{Q}}}$$

$\cos\Omega$ is independent of the particular set of vectors used to fix the k-spaces, only depending on these k-spaces. For, replacing one of the $a^i \dots q^i$ by a linear combination of them will merely multiply the determinants involving these components by a factor, which will appear once in both V_{a-q} and $\mathrm{S}_{a,\mathrm{A}}$ and hence cancel out. But by such

[1] For a simple proof of this formula for n-space see the Mathematical Gazette, Vol. X, 1921, p. 324.

replacements we can go from any set of vectors $a^i \ldots q^i$ to any other set determining the same k-space. Ω is the "angle of projectivity" between the k-spaces.

If we use the Cartesian coordinates previously introduced, which makes the first one of the coordinate k-spaces, the numerator of (18) will reduce to a single term, containing V_{a-q} as a factor, and on cancelling this out, we shall have :

$$(19) \qquad \cos\Omega = \frac{\begin{vmatrix} A^1 & A^2 & \ldots & A^k \\ \vdots & \vdots & & \vdots \\ Q^1 & Q^2 & \ldots & Q^k \end{vmatrix}}{V_{A-Q}}$$

which shows that $\cos\Omega$ is the ratio of the volume in the $a^i \ldots q^i$ $(1 \ldots k)$ space formed by projecting the volume V_{A-Q}, to this last volume ; where by projecting the volume we mean taking the components of the vectors used to form V_{A-Q} in the $a^i \ldots q^i$ space for the projected volume. Since we are using Cartesian coordinates, the denominator is ;

$$(20) \qquad V_{A-Q} = \sqrt{\sum \begin{vmatrix} A^i & A^j & \ldots & A^s \\ \vdots & \vdots & & \vdots \\ Q^i & Q^j & \ldots & Q^s \end{vmatrix}^2}$$

(Each combination counts only once in the summation.)

as follows from (15) since $g_{ij} = \delta_{ij}$. This shows that $\cos\Omega < 1$ also that if we formed expressions analogous to (19) for all the coordinate k-spaces ("direction cosines" of the k-space $A-Q$) the sum of their squares would be unity.

Finally, if we take two k-spaces given by sets of vectors of unit length $(m^1 \ldots m^k)$ and $(n^1 \ldots n^k)$, and determine the maximum or minimum values of the angle θ between two lines, one in each space ; using a method entirely analogous to that given by Ganguly,[1] we obtain the equation (in terms summed for i, $0 < i \leq n$):

$$(21) \qquad \begin{vmatrix} \cos\theta & m^1_i m^2_i \cos\theta & \ldots & m^1_i m^k_i \cos\theta & m^1_i n^1_i & \ldots & m^1_i n^k_i \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ m^k_i m^1_i \cos\theta & m^k_i m^2_i \cos\theta & \ldots & m^k_i m^k_i \cos\theta & m^k_i n^1_i & \ldots & m^k_i n^k_i \\ n^1_i m^1_i & n^1_i m^2_i & \ldots & n^1_i m^k_i & \cos\theta & \ldots & n^1 m^k \cos\theta \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ n^k_i m^1_i & n^k_i m^2_i & \ldots & n^k_i m^k_i & n^k_i n^1_i \cos\theta & \ldots & \cos\theta \end{vmatrix} = 0$$

[1] l. c., p. 16.

which shows that the product of the k values of $\cos^2\theta$ is:

$$(22) \qquad \frac{|\, m_i^s \; n_i^t \,|^2}{|\, m_i^s \; m_i^t \,| \; |\, n_i^s \; n_i^t \,|}$$

in which we have merely written the term in the sth row and tth column in each determinant. But in Cartesian coordinates, (17) gives: on replacing $a^i \ldots q^i$ by $m_i^1 \ldots m_i^k$ and $A^i \ldots Q^i$ by $n_i^1 \ldots n_i^k$:

$$(23) \qquad S_{m-n} = \sum |\, m_{j_t}^s \,| \; |\, n_{j_t}^s \,| = |\, m_i^s \; n_i^t \,|$$

where the summation extends over all combinations of k integers out of n $(j_1 \ldots j_k)$ each combination counted once, and the second equality follows from a well-known matrix identity.[1] This shows that (22) is equivalent to $S_{m-n}^2/V_m^2 V_n^2 = \cos^2\Omega$, and hence that the angle of projectivity of two k-spaces is the product of the k extremal values of $\cos\theta$, θ being the angle between a pair of lines one in each space.

We have thus shown that the invariant S, which is a function of two sets of k-vectors expressed by (17) in oblique coordinates, and by (23) in Cartesian coordinates is the generalization of the scalar product, being equal to the product of the volumes of the two k-parallelopipeds constructed on the two sets of vectors, by the cosine of the angle of projectivity of the two k-spaces determined by them. The cosine of the angle of projectivity of two k-spaces is equal to the ratio of the volume of a k-parallelopiped in one of the spaces to that of its projection in the other space: it is also equal to the product of k extremal values of the cosine of the angle between a pair of lines, one from each of the k-spaces.

[1] Scott and Mathews, Determinants, pp. 50-51.

On the motion of two Spheroids in an infinite liquid

BY

Nripendranath Sen, M.Sc.

The first writer to attempt the problem of the motion of two spheroids or ellipsoids in an infinite liquid is Prof. Karl Pearson[1] whose method does not, however, admit of further development and does not, therefore, lead to the complete solution of the problem. In a previous issue of this Bulletin,[2] Dr. Bibhutibhusan Datta attempted the problem of motion of two spheriods of small eccentricities in an infinite liquid along their common axis of revolution. In a recent issue of the American Journal of Mathematics,[3] Dr. Datta has solved the more general case of the same problem *viz.*, the motion of two spheroids of any eccentricities in an infinite liquid along their common axis of revolution.

The object of the present paper is to present the solution of a much more difficult problem *viz.*, the problem of the motion of an infinite liquid due to arbitrary movement of two speriods, both prolate and oblate, having a common axis of revolution. The problem has been completely solved first for two prolate spheriods, having any velocity of translation together with any velocity of rotation with no limitation regarding their ellipticities and central distance, and the solution for the case of two oblate speroids has been deduced therefrom by suitable substitutions.

I have shown that Dr. Datta's results of the problem referred to above may be deduced as a particular case of the general problem discussed in the present paper.

[1] Karl Pearson—" On the motion of spherical and ellipsoidal bodies in fluid media Part II. Quart. Journ. Math. Vol. 20.

[2] Bibhutibhushan Datta, D.Sc.—" On the motion of two spheroids in an infinite liquid along the common axis of revolution " Bul. Cal Math. Soc., Vol. 7, pp. 49-60.

[3] Bibhutibhushan Datta, D.Sc.—" On the motion of two spheroids in an infinite liquid along their common axis of revolution," Americ. Journ. Math., Vol. 43, pp. 31-42, 1921.

MOTION OF TWO PROLATE SPHEROIDS HAVING A COMMON AXIS OF REVOLUTION.

2. Let O_1, O_2 be centres of the two spheroids, $O_1\,O_2$ the common axis of revolution taken as z—axis and let the two systems of co-ordinates referred to parallel axes at O_1 and O_2 be

$$x_1=\kappa_1(1-\mu_1^2)^{\frac{1}{2}}\,(\lambda_1^2-1)^{\frac{1}{2}}\cos\omega\,;\qquad x_2=\kappa_2(1-\mu_2^2)^{\frac{1}{2}}\,(\lambda_2^2-1)^{\frac{1}{2}}\cos\omega$$

$$y_1=\kappa_1(1-\mu_1^2)^{\frac{1}{2}}\,(\lambda_1^2-1)^{\frac{1}{2}}\sin\omega\,;\qquad y_2=\kappa_2(1-\mu_2^2)^{\frac{1}{2}}\,(\lambda_2^2-1)^{\frac{1}{2}}\sin\omega$$

$$z_1=\kappa_1\mu_1\lambda_1\,;\qquad z_2=\kappa_2\mu_2\lambda_2$$

where (λ_1,μ_1,ω), (λ_2,μ_2,ω) are the two systems of plolate spheroidal coordinates so that $\lambda_1=\lambda_{10}$, $\lambda_2=\lambda_{20}$ on the surfaces of the given spheroids at O_1 and O_2 whose semi-axes are $a_1,c_1(a_1>c_1)$ and $a_2,c_2(a_2>c_2)$ respectively, whose eccentricities are e_1,e_2 respectively and central distance $O_1O_2=s$. Let (u_1,v_1,w_1,p_1,q_1) and (u_2,v_2,w_2,p_2,q_2) be the components of motion of the two spheroids, whose rotations about z—axis have not been taken into account here owing to the fact that there will be no motion of the liquid due to such rotations. To find the liquid motion due to such motions of the spheroids.

The problem before us is, therefore, to find a velocity potential ϕ satisfying the following conditions *viz.*,

$$\nabla^2\phi=0 \qquad \ldots \quad (1)$$

$$\phi=0 \text{ at infinity } i.e. \text{ when } \lambda=\infty \qquad \ldots \quad (2)$$

$$\begin{aligned}\frac{\partial\phi}{\partial\lambda_1}&=-\left\{\left(u_1\frac{\partial x_1}{\partial\lambda_1}+v_1\frac{\partial y_1}{\partial\lambda_1}+w_1\frac{\partial z_1}{\partial\lambda_1}\right)+p_1\left(y_1\frac{\partial z_1}{\partial\lambda_1}-z_1\frac{\partial y_1}{\partial\lambda_1}\right)\right.\\&\quad\left.+q_1\left(z_1\frac{\partial x_1}{\partial\lambda_1}-x_1\frac{\partial z_1}{\partial\lambda_1}\right)\right\}=-\kappa_1w_1P_1(\mu_1)-\left\{\kappa_1u_1\frac{\lambda_1P^1{}_1(\mu_1)}{(\lambda_1^2-1)^{\frac{1}{2}}}\right.\\&\quad\left.+\frac{\kappa_1^2q_1P_2^1(\mu_1)}{3(\lambda_1^2-1)^{\frac{1}{2}}}\right\}\cos\omega-\left\{\kappa_1v_1\frac{\lambda_1}{(\lambda_1^2-1)^{\frac{1}{2}}}P^1{}_1(\mu_1)\right.\\&\quad\left.-\frac{\kappa_1^2p_1}{3(\lambda_1^2-1)^{\frac{1}{2}}}P_2^1(\mu_1)\right\}\sin\omega \text{ when } \lambda_1=\lambda_{10}\end{aligned} \qquad \ldots \quad (3)$$

and
$$\begin{aligned}\frac{\partial\phi}{\partial\lambda}&=-\kappa_2\omega_2P_1(\mu_2)-\left\{\kappa_2u_2\frac{\lambda_2P^1{}_1(\mu_2)}{(\lambda_2^2-1)^{\frac{1}{2}}}\right.\\&\quad\left.+\frac{\kappa_2^2q_2P_2^1(\mu_2)}{3(\lambda_2^2-1)^{\frac{1}{2}}}\right\}\cos\omega-\left\{\kappa_2v_2\frac{\lambda_2P_1^1(\mu_2)}{(\lambda_2^2-1)^{\frac{1}{2}}}-\frac{\kappa_2^2p_2P_2^1(\mu_2)}{3(\lambda_2^2-1)^{\frac{1}{2}}}\right\}\sin\omega\end{aligned}$$

when
$$\lambda_2=\lambda_{20} \qquad \ldots \quad (4)$$

3. Assume

$$\phi=\sum_{n=1}^{\infty}\{A_nP_n(\mu_1)Q_n(\lambda_1)+a_nP_n(\mu_2)Q_n(\lambda_2)$$

$$+P_n^1(\mu_1)Q_n^1(\lambda_1)(B_n\cos\omega+C_n\sin\omega)$$

$$+P_n^1(\mu_2)Q_n^1(\lambda_2)(b_n\cos\omega+c_n\sin\omega)\} \qquad \dots \quad (5)$$

Evidently this value of ϕ satisfies (1) and (2). Now, to determine A's, B's, C's etc., so as to satisfy the boundary conditions (3) and (4).

It has been proved in a previous paper[1] of the author that

$$P_n^{\sigma}(\mu_2)Q_n^{\sigma}(\lambda_2)=(-)^{\sigma}\frac{\lfloor n+\sigma}{\lfloor n-\sigma}\sum_{m=\sigma}^{\infty}(2m+1)\frac{\lfloor m-\sigma}{\lfloor m+\sigma}\omega_1(m,n)$$

$$\times P_m^{\sigma}(\mu_1)P_m^{\sigma}(\lambda_1) \qquad \dots \quad (6)$$

for all +*ve* integral values of σ including zero,

$$\text{where } \omega_1(m,n)=\frac{(-)^n2^n\lfloor n}{\lfloor 2n+1}\rho_1^{n+1}\left[D_{t_1}^n+\frac{\rho_1^2}{2\cdot(2n+3)}D_{t_1}^{n+2}+\dots\right]Q_m(t_1)$$

$$\text{where} \qquad \rho_1^2=\frac{k_2^2}{k_1^2}=\frac{a_2^2e_2^2}{a_1^2e_1^2},\quad t_1=\frac{s}{k_1},\quad D_{t_1}^n=\frac{d^n}{dt_1^n} \qquad \dots \quad (7)$$

Substituting the value of $P_n^{\sigma}(\mu_2)Q_n^{\sigma}(\lambda_2)$ when $\sigma=0$, in (5) from (6) we have

$$\phi=\sum_{n=1}^{\infty}\Big\{A_nP_n(\mu_1)Q_n(\lambda_1)+a_n\sum_{m=1}^{\infty}(2m+1)\omega_1(m,n)P_m(\mu_1)P_m(\lambda_1)$$

$$+P_n^1(\mu_1)Q_n^1(\lambda_1)(B_n\cos\omega+C_n\sin\omega)$$

$$-(b_n\cos\omega+c_n\sin\omega)\sum_{m=1}^{\infty}\frac{n(n+1)(2m+1)}{m(m+1)}\omega_1(m,n)P_m^1(\mu_1)P_m^1(\lambda_1)\Big\}$$

[1] Nripendranath Sen—"On the steady motion of a viscous fluid due to the rotation of two spheroids." "Bul. Cal. Math. Soc." *Present issue*. Results (6), (8), (9), (10) and (11).

Hence from the surface condition (3), we have,

$$-k_1 w_1 P_1(\mu_1) - \left\{ k_1 u_1 \frac{\lambda_{10} P_1^1(\mu_1)}{(\lambda_{10}^2 - 1)^{\frac{1}{2}}} + \frac{k_1^2 q_1 P_2^1(\mu_1)}{3(\lambda_{10}^2 - 1)^{\frac{1}{2}}} \right\} \cos\omega$$

$$- \left\{ k_1 v_1 \frac{\lambda_{10} P_1^1(\mu_1)}{(\lambda_{10}^2 - 1)^{\frac{1}{2}}} - \frac{k_1^2 p_1 P_2^1(\mu_1)}{3(\lambda_{10}^2 - 1)^{\frac{1}{2}}} \right\} \sin\omega$$

$$= \sum_{n=1}^{\infty} \left\{ A_n P_n(\mu_1) Q'_n(\lambda_{10}) + a_n \sum_{m=1}^{\infty} (2m+1)\omega_1(m,n) P'_m(\lambda_{10}) P_m(\mu_1) \right.$$

$$+ P^1_n(\mu_1) Q'^1_n(\lambda_{10})(B_n \cos\omega + C_n \sin\omega)$$

$$\left. -(b_n \cos\omega + c_n \sin\omega) \sum_{m=1}^{\infty} \frac{n(n+1)(2m+1)}{m(m+1)} \omega_1(m,n) P'^1_m(\lambda_{10}) P^1_m(\mu_1) \right\}$$

at every point of the spheroid $\lambda_1 = \lambda_{10}$ *i.e.* for all values of μ_1 and ω.

$\therefore$ Equating the co-efficients of $\cos\omega$, $\sin\omega$, and co-efficients of P_n's and P_n^1's, we have

$$-k_1 w_1 = A_1 Q'_1(\lambda_{10}) + 3P'_1(\lambda_{10}) \sum_{n=1}^{\infty} a_n \omega_1(1,n) \quad \cdots \quad (8)$$

$$0 = A_p Q'_p(\lambda_{10}) + (2p+1) P'_p(\lambda_{10}) \sum_{n=1}^{\infty} a_n \omega_1(p,n)$$

$$(p = 2, 3, \cdots ad\ inf.) \quad \cdots \quad (9)$$

$$-k_1 u_1 \frac{\lambda_{10}}{(\lambda_{10}^2 - 1)^{\frac{1}{2}}} = B_1 Q'^1_1(\lambda_{10}) - \frac{3}{2} P'^1_1(\lambda_{10}) \sum_{n=1}^{\infty} n(n+1)\omega_1(1,n) b_n \quad (10)$$

$$-\frac{k_1^2 q_1}{3(\lambda_{10}^2 - 1)^{\frac{1}{2}}} = B_2 Q'^1_2(\lambda_{10}) - \frac{5}{6} P'^1_2(\lambda_{10}) \sum_{n=1}^{\infty} n(n+1)\omega_1(2,n) b_n \quad (11)$$

$$0 = B_p Q'^1_p(\lambda_{10}) - \frac{(2p+1) P'^1_p(\lambda_{10})}{p(p+1)} \sum_{n=1}^{\infty} n(n+1)\omega_1(p,n) b_n \quad (12)$$

$$(p = 3, 4, \ldots ad\ inf.)$$

$$-k_1 v_1 \frac{\lambda_{10}}{(\lambda_{10}^2 - 1)^{\frac{1}{2}}} = C_1 Q'^1_1(\lambda_{10}) - \tfrac{3}{2} P'^1_1(\lambda_{10}) \sum_{n=1}^{\infty} n(n+1)\omega_1(1,n) c_n \quad (13)$$

$$\frac{k_1^2 p_1}{3(\lambda_{10}^2-1)^{\frac{1}{2}}}=C_2 Q'^{1}_{2}(\lambda_{10})-\tfrac{5}{6}P'^{1}_{2}(\lambda_{10})\sum_{n=1}^{\infty} n(n+1)\omega_1(2,n)c_n \quad (14)$$

$$0=C_p Q'^{1}_{p}(\lambda_{10})-\frac{(2p+1)}{p(p+1)}P'^{1}_{p}(\lambda_{10})\sum_{n=1}^{\infty} n(n+1)\omega_1(p,n)c_n \ldots \quad (15)$$

$$(p=3,4,\ldots ad\ inf.)$$

The corresponding equations giving a's, b's, c's can be written down from symmetry from the above equations or from the boundary condition (4). Thus,

$$-k_2 w_2=a_1 Q'_1(\lambda_{20})+3\sum_{n=1}^{\infty} A_n\omega_2(1,n) \quad \ldots \quad (16)$$

$$0=a_p Q'_p(\lambda_{20})+(2p+1)P'_p(\lambda_{20})\sum_{n=1}^{\infty} A_n\omega_2(p,n) \quad (17)$$

$$(p=2,3\ldots ad\ inf.)$$

$$-k_2 u_2 \frac{\lambda_{20}}{(\lambda_{20}^2-1)^{\frac{1}{2}}}=b_1 Q'^{1}_{1}(\lambda_{20})-\tfrac{3}{2}P'^{1}_{1}(\lambda_{20})\sum_{n=1}^{\infty} n(n+1)\omega_2(1,n)B_n \quad (18)$$

$$-\frac{k_2^2 q_2}{3(\lambda_{20}^2-1)^{\frac{1}{2}}}=b_2 Q'^{1}_{2}(\lambda_{20})-\tfrac{5}{6}P'^{1}_{2}(\lambda_{20})\sum_{n=1}^{\infty} n(n+1)\omega_2(2,n)B_n \ldots \quad (19)$$

$$0=b_p Q'^{1}_{p}(\lambda_{20})-\frac{(2p+1)P'^{1}_{p}(\lambda_{20})}{p(p+1)}\sum_{n=1}^{\infty} n(n+1)\omega_2(p,n)B_n \quad (20)$$

$$(p=3,4\ldots ad\ inf.)$$

$$-k_2 v_2 \frac{\lambda_{20}}{(\lambda_{20}^2-1)^{\frac{1}{2}}}=c_1 Q'^{1}_{1}(\lambda_{20})-\tfrac{3}{2}P'^{1}_{1}(\lambda_{20})\sum_{n=1}^{\infty} n(n+1)\omega_2(1,n)C_n \quad (21)$$

$$\frac{k_2^2 p_2}{3(\lambda_{20}^2-1)^{\frac{1}{2}}}=c_2 Q'^{1}_{2}(\lambda_{20})-\tfrac{5}{6}P'^{1}_{2}(\lambda_{20})\sum_{n=1}^{\infty} n(n+1)\omega_2(2,n)C_n \quad (22)$$

$$0=c_p Q'^{1}_{p}(\lambda_{20})-\frac{(2p+1)}{p(p+1)}P'^{1}_{p}(\lambda_{20})\sum_{n=1}^{\infty} n(n+1)\omega_2(p,n)C_n \ldots \quad (23)$$

$$(p=3,4,\ldots ad\ inf.)$$

where $\omega_2(m,n) = \dfrac{(-)^n 2^n \lfloor n}{\lfloor 2n+1} \rho_2^{n+1} \Big[D^n_{t_2}$

$$+ \frac{\rho_2^2}{2\cdot(2n+3)} D^{n+2}_{t_2} + \text{etc.} \Big] P_m(t_2) \quad \dots \quad (24)$$

where $\qquad \rho_2 = \dfrac{k_1}{k_0},\ t_2 = \dfrac{s}{k_0},\ D^n_{t_2} = \dfrac{d^n}{dt_2^n}$ etc

The equations (8) to (24) are sufficient to determine sets of unknown constants A's, B's, C's etc., as will be shewn presently.

To determine A's and a's, subtitute the values of a's in (8) and (9) from (16) and (17). We, then, have after a little simplification,

$$A_1 - \sum_{n=1}^{\infty} \theta_{1n} A_n = -\frac{k_1 w_1}{Q'_1(\lambda_{10})} + \frac{3\omega_1(1,1)}{Q'_1(\lambda_{10})Q'_1(\lambda_{20})} k_2 \omega_2 \quad \dots \quad (25)$$

$$A_p - \sum_{n=1}^{\infty} \theta_{pn} A_n = (2p+1)\frac{P'_p(\lambda_{10})\omega_1(p,1)}{Q'_p(\lambda_{10})Q'_1(\lambda_{10})} k_2 \omega_2 \quad \dots \quad (26)$$

$$(p=2,3,\dots ad\ inf.)$$

where $\theta_{pn} = (2p+1)\dfrac{P'_p(\lambda_{10})}{Q'_p(\lambda_{10})} \sum_{m=1}^{\infty} (2m+1) \dfrac{P'_m(\lambda_{20})}{Q'_m(\lambda_{20})} \omega_1(p,m)\omega_2(m,n)$ (27)

The equations giving a's can be found out independently or may be written from (25), (26) etc. Thus from symmetry,

$$a_1 - \sum_{n=1}^{\infty} \theta'_{1n} a_n = -\frac{k_2 w_2}{Q'_1(\lambda_{20})} + \frac{3\omega_2(1,1)}{Q'_1(\lambda_{20})Q'_1(\lambda_{10})} k_1 w_1 \quad \dots \quad (28)$$

$$a_p - \sum_{n=1}^{\infty} \theta'_{pn} a_n = (2p+1)\frac{P'_p(\lambda_{20})\omega_2(p,1)}{Q'_p(\lambda_{20})Q'_1(\lambda_{10})} k_1 w_1 \quad \dots \quad (29)$$

$$(p=2,3,\dots ad\ inf.)$$

where $\theta'_{pn} = (2p+1)\dfrac{P'_p(\lambda_{20})}{Q'_p(\lambda_{20})} \sum_{m=1}^{\infty} (2m+1) \dfrac{P'_m(\lambda_{10})}{Q'_m(\lambda_{10})} \omega_2(p,m)\omega_1(m,n)$

(30)

To find B's and b's, substitute the values of b's in (10) and (11) from (18), (19) and (20), we have, after a little simplification,

$$B_1-\sum_{n=1}^{\infty}\phi_{1n}B_n=-\frac{\lambda_{10}k_1u_1}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{10})}$$

$$-\frac{3P'^1_1(\lambda_{10})\lambda_{20}\omega_1(1,1)k_2u_2}{Q'^1_1(\lambda_{10})Q'^1_1(\lambda_{20})(\lambda_{20}^2-1)^{\frac{1}{2}}}-\frac{3P'^1_1(\lambda_{10})\omega_1(1,2)k_2^2q_2}{Q'^1_1(\lambda_{10})(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{20})}\tag{31}$$

$$B_2-\sum_{n=1}^{\infty}\phi_{2n}B_n=-\frac{k_1^2q_1}{3(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{10})}$$

$$\frac{5}{3}\frac{P'^1_2(\lambda_{10})\lambda_{20}\omega_1(2,1)k_2u_2}{Q'^1_2(\lambda_{10})(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{20})}-\frac{5}{3}\frac{P'^1_2(\lambda_{10})\omega_1(2,2)k_2^2q_2}{Q'^1_2(\lambda_{10})(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{20})}\tag{32}$$

$$B_p-\sum_{n=1}^{\infty}\phi_{pn}B_n=-\frac{2(2p+1)P'^1_p(\lambda_{10})}{p(p+1)Q'^1_p(\lambda_{10})}\times\left[\frac{\lambda_{20}\omega_1(p,1)k_2u_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{20})}+\frac{\omega_1(p,2)k_2^2q_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{20})}\right]\dots\tag{33}$$

$(p=3,4\cdots ad\ inf)$

where $$\phi_{pn}=\frac{(2p+1)P'^1_p(\lambda_{10})}{p(p+1)Q'^1_p(\lambda_{10})}\sum_{m=1}^{\infty}n(n+1)(2m+1)\omega_1(p,m)\omega_2(m,n)\frac{P'^1_m(\lambda_{20})}{Q'^1_m(\lambda_{20})}\quad\dots\tag{34}$$

Also from symmetry,

$$b_1-\sum_{n=1}^{\infty}\phi'_{1n}b_n=-\frac{\lambda_{20}k_2u_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{20})}-\frac{3P'^1_1(\lambda_{20})\lambda_{10}\omega_2(1,1)k_1u_1}{Q'^1_1(\lambda_{10})Q'^1_1(\lambda_{20})(\lambda_{10}^2-1)^{\frac{1}{2}}}$$

$$-\frac{3P'^1_1(\lambda_{20})\omega_2(1,2)k_1^2q_1}{Q'^1_1(\lambda_{20})Q'^1_2(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}}\quad\dots\tag{35}$$

$$b_2-\sum_{n=1}^{\infty}\phi'_{2n}b_n=-\frac{k_2^2q_2}{3(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{20})}$$

$$-\frac{5}{3}\frac{P'^1_2(\lambda_{20})\lambda_{10}\omega_2(2,1)k_1u_1}{Q'^1_2(\lambda_{20})(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{10})}-\frac{5}{3}\frac{P'^1_2(\lambda_{20})\omega_2(2,2)k_1^2q_1}{Q'^1_2(\lambda_{10})Q'^1_2(\lambda_{20})(\lambda_{10}^2-1)^{\frac{1}{2}}}\tag{36}$$

$$b_p-\sum_{n=1}^{\infty}\phi'_{pn}b_n=-\frac{2(2p+1)}{p(p+1)}\frac{P'^{\frac{1}{2}}_p(\lambda_{20})}{Q'^{\frac{1}{2}}_p(\lambda_{20})}\left[\frac{\lambda_{10}\omega_2(p,1)k_1u_1}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^{\frac{1}{2}}_1(\lambda_{10})}\right.$$

$$\left.+\frac{\omega_2(p,2)k_1^2q_1}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^{\frac{1}{2}}_2(\lambda_{10})}\right] \quad \ldots \quad (37)$$

$$(p=3,4,\ldots ad\ inf.)$$

where $\phi'_{pn}=\dfrac{n(n+1)(2p+1)}{p(p+1)}\dfrac{P'^{\frac{1}{2}}_p(\lambda_{20})}{Q'^{\frac{1}{2}}_p(\lambda_{20})}$

$$\sum_{m=1}^{\infty}(2m+1)\omega_2(p,m)\omega_1(m,n)\frac{P'^{\frac{1}{2}}_m(\lambda_{10})}{Q'^{\frac{1}{2}}_m(\lambda_{10})} \quad \ldots \quad (38)$$

Proceeding in a similar way, we obtain the equations giving C's and c's. Thus,

$$C_1-\sum_{n=1}^{\infty}\phi_{1n}C_n=-\frac{\lambda_{10}k_1v_1}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^{\frac{1}{2}}_1(\lambda_{10})}-\frac{3P'^{\frac{1}{2}}_1(\lambda_{10})\lambda_{20}\omega_1(1,1)k_2v_2}{Q'^{\frac{1}{2}}_1(\lambda_{10})Q'^{\frac{1}{2}}_1(\lambda_{20})(\lambda_{20}^2-1)^{\frac{1}{2}}}$$

$$+\frac{3P'^{\frac{1}{2}}_1(\lambda_{10})\omega_1(1,2)k_2^2p_2}{Q'^{\frac{1}{2}}_1(\lambda_{10})(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^{\frac{1}{2}}_2(\lambda_{20})} \quad \ldots \quad (39)$$

$$C_2-\sum_{n=1}^{\infty}\phi_{2n}C_n=\frac{k_1^2p_1}{3(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^{\frac{1}{2}}_2(\lambda_{10})}$$

$$-\frac{5}{3}\frac{P'^{\frac{1}{2}}_2(\lambda_{10})\lambda_{20}\omega_1(2,1)k_2v_2}{Q'^{\frac{1}{2}}_2(\lambda_{10})Q'^{\frac{1}{2}}_1(\lambda_{20})(\lambda_{20}^2-1)^{\frac{1}{2}}}+\frac{5}{3}\frac{P'^{\frac{1}{2}}_2(\lambda_{10})\omega_1(2,2)k_2^2p_2}{Q'^{\frac{1}{2}}_2(\lambda_{10})(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^{\frac{1}{2}}_2(\lambda_{20})}$$

(40)

$$C_p-\sum_{n=1}^{\infty}\phi_{pn}C_n=-\frac{2(2p+1)}{p(p+1)}\frac{P'^{\frac{1}{2}}_p(\lambda_{10})}{Q'^{\frac{1}{2}}_p(\lambda_{10})}\times\left[\frac{\lambda_{20}\omega_1(p,1)k_2v_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^{\frac{1}{2}}_1(\lambda_{20})}\right.$$

$$\left.-\frac{\omega_1(p,2)k_2^2p_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^{\frac{1}{2}}_2(\lambda_{20})}\right] \ldots \quad (41)$$

$$(p=3,4,\ldots ad\ inf.)$$

Also,

$$c_1-\sum_{n=1}^{\infty}\phi'_{1n}c_n=-\frac{\lambda_{20}k_2v_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^{\frac{1}{2}}_1(\lambda_{20})}-\frac{3P'^{\frac{1}{2}}_1(\lambda_{20})\lambda_{10}\omega_2(1,1)k_1v_1}{Q'^{\frac{1}{2}}_1(\lambda_{20})Q'^{\frac{1}{2}}_1(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}}$$

$$+\frac{3P'^{\frac{1}{2}}_1(\lambda_{20})\omega_2(1,2)k_1^2p_1}{Q'^{\frac{1}{2}}_1(\lambda_{20})(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^{\frac{1}{2}}_2(\lambda_{10})} \quad \ldots \quad (42)$$

$$c_2 - \sum_{n=1}^{\infty} \phi'_{2n} c_n = \frac{k_2^2 p_2}{3(\lambda_{20}^2-1)^{\frac{1}{2}} Q'^1_2(\lambda_{20})} - \frac{5}{3}\,\frac{P'^1_2(\lambda_{20})\lambda_{10}\omega_2(2,1)k_1 v_1}{Q'^1_2(\lambda_{20})Q'^1_1(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}}$$

$$+\frac{5}{3}\,\frac{P'^1_2(\lambda_{20})\omega_2(2,2)k_1^2 p_1}{Q'^1_2(\lambda_{20})Q'^1_2(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}} \quad \ldots \quad (43)$$

$$c_p - \sum_{n=1}^{\infty} \phi'_{pn} c_n = -\frac{2(2p+1)}{p(p+1)}\,\frac{P'^1_p(\lambda_{20})}{Q'^1_p(\lambda_{20})} \times \left[\frac{\lambda_{10}\omega_2(p,1)k_1 v_1}{(\lambda_{10}^2-1)^{\frac{1}{2}} Q'^1_1(\lambda_{10})} \right.$$

$$\left. - \frac{\omega_2(p,2)k_1^2 p_1}{(\lambda_{10}^2-1)^{\frac{1}{2}} Q'^1_2(\lambda_{10})} \right] \quad \ldots \quad (44)$$

$$(p=3,4 \ldots ad\ inf\,)$$

where ϕ_{pn}, ϕ'_{pn} are defined as in (34) and (39).

Each of the above sets of equations giving A's, a's, B's, b's, C's, c's may be solved determinantally and it is plain that A's and a's are linear in w_1 and w_2 B's and b's are linear in u_1, u_2, q_1 and q_2, and C's and c's are linear in v_1, v_2 p_1 and p_2. The theory of solutions of such equations has been worked out by Hill, Poincare, Von. Koch, Teoplitz, Hilbert and others. The constants can, therefore, be determined and the problem solved.

4. The complete algebraic values of A's and B's and other constants thus determined are not much suitable for numerical calculations. But the constants may be calculated to any degree of approximation as follows :—

From (7)

$$\omega_1(m,n) = (-)^n \frac{2^n \lfloor n}{\lfloor 2n+1} \rho_1^{n+1} \left[D^n_{t_1} + \frac{\rho_1^2}{2\cdot(2n+3)} D^{n+2}_{t_1} + \text{etc.} \right] Q_m(t_1)$$

where $\rho_1 = \dfrac{k_2}{k_1}$, $t_1 = \dfrac{s}{k_1}$

$$= \frac{2^{m+n} \lfloor m \lfloor n \lfloor m+n}{\lfloor 2m+1 \lfloor 2n+1} \frac{k_1^m k_2^{n+1}}{s^{m+n+1}} \left[1 + \frac{\lfloor m+n+2}{2 \lfloor m+n} \frac{1}{s^2} \left(\frac{k_1^2}{2m+3} + \frac{k_2^2}{2m+3} \right) \right.$$

$$+ \frac{\lfloor m+n+4}{2\cdot 4\cdot \lfloor m+n} \frac{1}{s^4} \left\{ \frac{k_1^4}{(2m+3)(2m+5)} + \frac{2k_1^2 k_2^2}{(2m+3)(2n+3)} \right.$$

$$\left. \left. + \frac{k_2^4}{(2n+3)(2n+5)} \right\} + \text{etc.} \right] \quad \ldots \quad (45)$$

substituting the values of $Q_m(t_1)$ and simplifying.

Thus the lowest order of ω_1 (m, n) is $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^{m+n+1}$ and such is the case in ω_2 (m, n). Hence, from (27), the lowest value of

$$\theta_{pn}=3\,(2p+1)\,\frac{P'_p\,(\lambda_{10})\,\omega_1\,(p,1)\,\omega_2\,(1,n)}{Q'_p\,(\lambda_{10})\,Q'_1\,(\lambda_{20})}\text{ taking } m=1$$

$$=\frac{2^{p+n}\,\underline{|p}\,\underline{|n}\,\underline{|p+1}\,\underline{|n+1}}{3\,\underline{|2p}\,\underline{|2n+1}}\;\frac{P'_p\,(\lambda_{10})\,k_1{}^{p+n+1}\,k_2{}^3}{Q'_p\,(\lambda_{10})\,Q'_1\,(\lambda_{20})\,s^{p+n+4}}\quad\ldots\quad(46)$$

The lowest value of θ'_{pn} is obtained from (46) by interchanging k_1 and k_2, λ_{10} and λ_{20}.

From (34), the lowest value of ϕ_{pn}

$$=\frac{3\,(2p+1)\,n\,(n+1)\,P'_p{}^1\,(\lambda_{10})}{p\,(p+1)\,Q'_p{}^1\,(\lambda_{10})}\;\omega_1\,(p,1)\;\omega_2\,(1,n)\;\frac{P'_1{}^1\,(\lambda_{20})}{Q'_1{}^1\,(\lambda_{20})}$$

$$=\frac{2^{n+p}\,n\,(n+1)\,\underline{|p-1}\,\underline{|p}\,\underline{|n}\,\underline{|n+1}}{3\,\underline{|2p}\,\underline{|2n+1}}\;\frac{P'_p{}^1\,(\lambda_{10})\,P'_1{}^1\,(\lambda_{20})}{Q'_p{}^1\,(\lambda_{10})\,Q'_1{}^1\,(\lambda_{20})}$$

$$\frac{k_1{}^{n+p+1}\,k_2{}^3}{s^{n+p+4}}\quad\ldots\quad(47)$$

The lowest value of ϕ'_{pn} is obtained from (47) by writing k_1 for k_2 and *vice versa*, and interchanging λ_{10} and λ_{20}.

5. (A). If the spheroids are so separated that we can neglect the terms of the order $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^3$ and higher powers, we have, from (25) to (29),

$$A_1=-\frac{k_1\,w_1}{Q'_1\,(\lambda_{10})}\,,\;a_1=-\frac{k_2\,w_2}{Q'_1\,(\lambda_{20})}$$

A_p etc.$=0$, $a_p=0$, $(p=2,3\ldots ad.\ inf.)$

From (31) to (37),

$$B_1=-\frac{\lambda_{10}\,k_1\,u_1}{(\lambda_{10}{}^2-1)^{\frac{1}{2}}\,Q'_1{}^1\,(\lambda_{10})}\;;\;B_2=-\frac{k_1{}^2\,q_1}{3\,(\lambda_{10}{}^2-1)^{\frac{1}{2}}\,Q'_2{}^1\,(\lambda_{10})}$$

$B_p=0$, $(p=3,4\ldots ad.\ inf.)$

$$b_1 = -\frac{\lambda_{20}\, k_2\, u_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}\, Q'^1_1\,(\lambda_{20})}\,,\quad b_2 = -\frac{k_2^2\, q_2}{3\,(\lambda_{20}^2-1)^{\frac{1}{2}}\, Q'^1_2\,(\lambda_{20})}$$

$$b_p = 0, \qquad (p=3,4\ldots ad.\ inf.)$$

From (39) to (44),

$$C_1 = -\frac{\lambda_{10}\, k_1\, v_1}{(\lambda_{10}^2-1)^{\frac{1}{2}}\, Q'^1_1\,(\lambda_{10})}\,,\quad C_2 = \frac{k_1^2\, p_1}{3\,(\lambda_{10}^2-1)^{\frac{1}{2}}\, Q'^1_2\,(\lambda_{10})}$$

$$C_p = 0 \qquad (p=3,4\ldots ad.\ inf.)$$

$$c_1 = -\frac{\lambda_{20}\, k_2\, v_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}\, Q'^1_1\,(\lambda_{20})}\,;\quad c_2 = \frac{k_2^2\, p_2}{3\,(\lambda_{20}^2-1)^{\frac{1}{2}}\, Q'^1_2\,(\lambda_{20})}$$

$$c_p = 0 \qquad (p=3,4\ldots ad.\ inf.)$$

Hence, from (5) $\phi = A_1\, P_1\,(\mu_1)\, Q_1\,(\lambda_1) + a_1\, P_1\,(\mu_2)\, Q_1\,(\lambda_2)$

$$+P_1^1\,(\mu_1)\, Q_1^1\,(\lambda_1)\,(B_1 \cos\omega + C_1 \sin\omega)$$

$$+P_2^1\,(\mu_1)\, Q_2^1\,(\lambda_1)\,(B_2 \cos\omega + C_2 \sin\omega)$$

$$+P_1^1\,(\mu_2)\, Q_1^1\,(\lambda_2)\,(b_1 \cos\omega + c_1 \sin\omega)$$

$$+P_2^1\,(\mu_2)\, Q_2^1\,(\lambda_2)\,(b_2 \cos\omega + c_2 \sin\omega) \qquad \ldots \quad (48)$$

where A_1, a_1 ; B_1, B_2, b_1. b_2 ; C_1, C_2, c_1, c_2 are given above.

(B) If we neglect terms of the order $\left(\dfrac{\text{linear dimension}}{\text{central distance}}\right)^5$ and higher powers, then we have from (25)

$$A_1 = -\frac{k_1\, w_1}{Q'_1\,(\lambda_{10})} + \frac{2k_1\, k_2^3\, w_2}{3\,Q'_1(\lambda_{10})\, Q'_1(\lambda_{20})}\,\frac{1}{s^3}\left[\because \omega_1\,(1,1) = \tfrac{2}{9}\frac{k_1\, k_2^2}{s^3}\right]$$

From (28) or from symmetry,

$$a_1 = -\frac{k_2\, w_2}{Q'_1\,(\lambda_{20})} + \tfrac{2}{3}\,\frac{k_2\, k_1^3\, w_1}{Q'_1\,(\lambda_{10})\, Q'_2\,(\lambda_{20})}\cdot\frac{1}{s^3}$$

From (26) $A_2 = 5\,\dfrac{P'_2\,(\lambda_{10})\,\omega_1\,(2,1)}{Q'_2\,(\lambda_{10})\, Q'_1\,(\lambda_{20})}\; k_2\, w_2$

$$= \tfrac{2}{3}\,\frac{P'_2\,(\lambda_{10})\, k_1^2\, k_2^3\, w_2}{Q'_2\,(\lambda_{10})\, Q'_1\,(\lambda_{20})\, s^4}\left[\because \omega_1\,(2,1) = \tfrac{2}{15}\,\frac{k_1^2\, k_2^2}{s^4}\right]$$

$$A_p = 0, \qquad (p=3,4,\ldots ad.\ inf.)$$

From (29),

$$a_2=\frac{2}{3}\,\frac{P'_2(\lambda_{20})\,w_1}{Q'_2(\lambda_{20})\,Q'_1(\lambda_{10})}\,\frac{k_1{}^3\,k_2{}^2}{s^4}$$

$$a_p=0 \qquad (p=3,4,\ldots ad.\ inf.)$$

Also from (31), (32), (33),

$$B_1=-\frac{\lambda_{10}\,k_1\,u_1}{(\lambda_{10}{}^2-1)^{\frac{1}{2}}Q'_1{}^1(\lambda_{10})}-\frac{2}{3}\,\frac{P'_1{}^1(\lambda_{10})\,\lambda_{20}\,k_2\,u_2\,k_1\,k^2{}_2}{Q'_1{}^1(\lambda_{10})\,Q'_1{}^1(\lambda_{20})(\lambda_{20}{}^2-1)^{\frac{1}{2}}s^3}$$

$$-\frac{2}{5}\,\frac{P'_1{}^1(\lambda_{10})\,q_2}{Q'_1{}^1(\lambda_{10})\,(\lambda_{20}{}^2-1)^{\frac{1}{2}}\,Q'_2{}^1(\lambda_{20})}\,\frac{k_1\,k_2{}^5}{s^4}$$

$$B_2=-\frac{k_1{}^2\,q_1}{3\,(\lambda_{10}{}^2-1)^{\frac{1}{2}}\,Q'_2{}^1(\lambda_{10})}$$

$$-\frac{2}{9}\,\frac{P'_2{}^1(\lambda_{10})\,\lambda_{20}\,k_2\,u_2}{Q'_2{}^1(\lambda_{10})\,(\lambda_{20}{}^2-1)^{\frac{1}{2}}\,Q'_1{}^1(\lambda_{20})}\cdot\frac{k_1{}^3\,k_2{}^2}{s^4}$$

$$B_p=0, \qquad (p=3,4,\ldots ad.\ inf.)$$

Similarly, $b_1=-\dfrac{\lambda_{20}\,k_2\,u_2}{(\lambda_{20}{}^2-1)^{\frac{1}{2}}\,Q'_1{}^1(\lambda_{20})}$

$$-\frac{2}{3}\,\frac{P'_1{}^1(\lambda_{20})\,\lambda_{10}\,k_1\,u_1}{Q'_1{}^1(\lambda_{20})\,Q'_1{}^1(\lambda_{10})\,(\lambda_{10}{}^2-1)^{\frac{1}{2}}}\cdot\frac{k_2\,k_1{}^2}{s^3}$$

$$-\frac{2}{5}\,\frac{P'_1{}^1(\lambda_{20})\,q_1\,k_1\,k_2}{Q'_1{}^1(\lambda_{20})\,(\lambda_{10}{}^2-1)^{\frac{1}{2}}\,Q'_2{}^1(\lambda_{10})}\,\frac{k_1{}^4}{s_4}.$$

$$b_2=-\frac{k_2^2q_2}{3(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{20})}-\frac{2}{9}\,\frac{P'^1_2(\lambda_{20})\lambda_{10}k_1u_1}{Q'^1_2(\lambda_{20})(\lambda_{10}^2-1)^{\frac{1}{2}}Q'(\lambda_{10})}\,\frac{k_1^3k_2^2}{s^4}$$

$$b_p=0, \qquad (p=3,4,\ldots ad\ inf.)$$

From (39), (40), (41), we have,

$$C_1=-\frac{\lambda_{10}k_1v_1}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{10})}-\frac{2}{3}\,\frac{P'^1_2(\lambda_{10})\lambda_{20}k_2v_2}{Q'^1_1(\lambda_{10})Q'^1_1(\lambda_{20})(\lambda_{20}^2-1)^{\frac{1}{2}}}\,\frac{k_1k_2^2}{s^3}$$

$$+\frac{2}{5}\,\frac{P'^1_1(\lambda_{10})p_2k_1k_2}{Q'^1_1(\lambda_{10})(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{20})}\,\frac{k_2^4}{s^4}$$

$$C_2 = \frac{k_1^2 p_1}{3(\lambda_{10}^2-1)^{\frac{1}{2}} Q'^1_2(\lambda_{10})} - \frac{2}{9} \frac{P'^1_2(\lambda_{10})\lambda_{20} k_2 v_2}{Q'^1_2(\lambda_{10}) Q'^1_1(\lambda_{20})(\lambda_{20}^2-1)^{\frac{1}{2}}} \frac{k_1^2 k_2^2}{s^4}$$

$$C_p = 0, \qquad (p=3,4,5 \ldots ad\ inf.)$$

Also,

$$c_1 = -\frac{\lambda_{20} k_2 v_2}{(\lambda_{20}^2-1)^{\frac{1}{2}} Q'^1_1(\lambda_{20})} - \frac{2}{3} \frac{P'^1_1(\lambda_{20})\lambda_{10} k_1 v_1}{Q'^1_1(\lambda_{20}) Q'^1_1(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}} \frac{k_2 k_1^2}{s^3}$$

$$+\frac{2}{5} \frac{P'^1_1(\lambda_{20}) k_1 k_2 p_1}{Q'^1_1(\lambda_{20}) Q'^1_2(\lambda_{10})(\lambda_{20}^2-1)^{\frac{1}{2}}} \frac{k_1^4}{s^4}$$

$$c_2 = \frac{k_2^2 p_2}{3(\lambda_{20}^2-1)^{\frac{1}{2}} Q'^1_2(\lambda_{20})} - \frac{2}{9} \frac{P'^1_2(\lambda_{20})\lambda_{10} k_1 v_1}{Q'^1_2(\lambda_{20}) Q'^1_1(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}} \frac{k_1^2 k_2^2}{s^4}$$

$$c_p = 0, \qquad (p=3,4,5 \ldots ad\ inf.)$$

Hence from (5),

$$\phi = \sum_{n=1}^{\infty} \{A_n P_n(\mu_1) Q_n(\lambda_1) + a_n P_n(\mu_2) Q_n(\lambda_2)$$

$$+ P^1_n(\mu_1) Q^1_n(\lambda_1)(B_n \cos\omega + C_n \sin\omega)$$

$$+ P^1_n(\mu_2) Q^1_n(\lambda_2)(b_n \cos\omega + c_n \sin\omega)\} \quad \ldots \quad (49)$$

where the constants are given above.

(C) If we neglect terms of the order $\left(\frac{\text{linear dimension}}{\text{central distance}}\right)^7$ and highers, powers we have from (25) and (45),

$$A_1 = -\frac{k_1 w_1}{Q'_1(\lambda_{10})}\left\{1+\theta_{11}\right\} + \frac{3\omega_1(1,1) k_2 w_2}{Q'_1(\lambda_{10}) Q'_1(\lambda_{20})}$$

$$= -\frac{k_1 w_1}{Q'_1(\lambda_{10})}\left\{1+\frac{4}{9} \frac{1}{Q'_1(\lambda_{10}) Q'_1(\lambda_{20})} \frac{k_1^3 k_2^3}{s^6}\right\}.$$

From (26),

$$A_2 = \frac{2}{3} \frac{P'_2(\lambda_{10}) k_2 w_2}{Q'_2(\lambda_{10}) Q'_1(\lambda_{20})} \frac{k_1^2 k_2^2}{s^4}\left\{1+\frac{10}{s^2}\left(\frac{k_1^2}{7}+\frac{k_2^2}{5}\right)\right\}$$

$$A_3 = 7 \frac{P'_3(\lambda_{10})}{Q'_3(\lambda_{10})} \frac{\omega_1(3,1)}{Q'_1(\lambda_{20})} k_2 w_2 = \frac{8}{15} \frac{P'_3(\lambda_{10}) k_2 w_2}{Q_3(\lambda_{10}) Q'_1(\lambda_{20})}; \frac{k_1^3 k_2^2}{s^5}$$

$$\left[\because \omega_1(3,1) = \frac{8}{105} \frac{k_1^3 k_2^2}{s^5}\right]$$

$$A_4 = 9\,\frac{P'_4(\lambda_{10})\omega_1(4,1)}{Q'_4(\lambda_{10})Q'_1(\lambda_{20})}\,k_2 w_2$$

$$= \frac{8}{21}\,\frac{P'_4(\lambda_{10})k_2 w_2}{Q'_4(\lambda_{10})Q'_1(\lambda_{20})}\,\frac{k_1^4 k_2^2}{s^6}\left[\because \omega_1(4,1) = \frac{8}{189}\,\frac{k_1^4 k_2^2}{s^6}\right]$$

$$A_p = 0, \qquad (p = 5, 6 \ldots ad\ inf.)$$

Similarly,

$$a_1 = -\frac{k_2 w_2}{Q'_1(\lambda_{20})}\left\{1 + \frac{4}{9}\,\frac{1}{Q'_1(\lambda_{10})Q'(\lambda_{20})}\,\frac{k_1^3 k_2^3}{s^6}\right\}$$

$$+\frac{2}{3}\,\frac{k_1 w_1}{Q'_1(\lambda_{10})Q'_1(\lambda_{20})}\,\frac{k_2 k_1^2}{s^3}\left\{1 + \frac{6}{5}\,\frac{k_1^2 + k_2^2}{s^2}\right\}$$

$$a_2 = \frac{2}{3}\,\frac{P'_2(\lambda_{20})k_1 w_1}{Q'_2(\lambda_{20})Q'_1(\lambda_{10})}\,\frac{k_1^2 k_2^2}{s^4}\left\{1 + \frac{10}{s^2}\left(\frac{k_2^2}{7} + \frac{k_1^2}{5}\right)\right\}$$

$$a_3 = \frac{8}{15}\,\frac{P'_3(\lambda_{20})k_1 w_1}{Q'_3(\lambda_{20})Q'_1(\lambda_{10})}\,\frac{k_2^3 k_1^2}{s^6}$$

$$a_4 = \frac{8}{21}\,\frac{P'(\lambda_{20})k_1 w_1}{Q'_4(\lambda_{20})Q'_1(\lambda_{10})}\,\frac{k_1^2 k_2^4}{s^6}$$

$$a_p = 0, \qquad (p = 5, 6, \ldots ad\ inf.)$$

From (31),

$$B_1 = -\frac{\lambda_{10} k_1 u_1}{(\lambda_{10}^2 - 1)^{\frac{1}{2}} Q_1'^1(\lambda_{10})}\left(1 + \phi_{11}\right)$$

$$-\frac{2}{3}\,\frac{P_1'^1(\lambda_{10})\lambda_{20} k_2 u_2}{Q_1'^1(\lambda_{10})Q_1'^1(\lambda_{20})(\lambda_{20}^2 - 1)^{\frac{1}{2}}}\,\frac{k_1 k_2^2}{s^3}\left(1 + \frac{6}{5}\,\frac{k_1^2 + k_2^2}{s^2}\right)$$

$$-\frac{2}{5}\,\frac{P_1'^1(\lambda_{10})k_1 k_2 q_2}{Q_1'^1(\lambda_{10})Q_2'^1(\lambda_{20})(\lambda_{20}^2 - 1)^{\frac{1}{2}}}\,\frac{k_2^4}{s^4}\left\{1 + \frac{10}{s^2}\left(\frac{k_1^2}{5} + \frac{k_2^2}{7}\right)\right\}$$

$$= -\frac{\lambda_{10} k_1 u_1}{(\lambda_{10}^2 - 1)^{\frac{1}{2}} Q_1'^1(\lambda_{10})}\left(1 + \frac{4}{9}\,\frac{P_1'^1(\lambda_{10})P_1'^1(\lambda_{20})}{Q_1'^1(\lambda_{10})Q_1'^1(\lambda_{20})}\,\frac{k_1^3 k_2^3}{s^6}\right)$$

$$-\frac{2}{3}\,\frac{P_1'^1(\lambda_{10})\lambda_{20} k_2 u_2}{Q_1'^1(\lambda_{10})Q_1'^1(\lambda_{20})(\lambda_{20}^2 - 1)^{\frac{1}{2}}}\,\frac{k_1 k_2^2}{s^3}\left(1 + \frac{6}{5}\,\frac{k_1^2 + k_2^2}{s^2}\right)$$

$$-\frac{2}{5}\,\frac{P_1'^1(_{10})k_1 k_2 q_2}{Q_1'^1(\lambda_{10})Q_2'^1(\lambda_{20})(\lambda_{20}^2 - 1)^{\frac{1}{2}}}\,\frac{k_2^4}{s_2^4}\left\{1 + \frac{10}{s^2}\left(\frac{k_1^2}{5} + \frac{k_2^2}{7}\right)\right\}$$

$$B_2=-\frac{k_1^2q_1}{3(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{10})}$$

$$-\frac{2}{9}\frac{P'^1_2(\lambda_{10})\lambda_{20}k_2u_2}{Q'^1_2(\lambda_{10})Q'^1_1(\lambda_{20})(\lambda_{20}^2-1)^{\frac{1}{2}}}\frac{k_1^2k_2^2}{s^4}\left\{1+\frac{10}{s^2}\left(\frac{k_1^2}{7}+\frac{k_2^2}{5}\right)\right\}$$

$$-\frac{8}{45}\frac{P'^1_2(\lambda_{10})k_2^2q_2}{Q'^1_2(\lambda_{10})Q'^1_2(\lambda_{20})(\lambda_{20}^2-1)^{\frac{1}{2}}}\frac{k_1^2k_2^3}{s^5}$$

$$B_3=-\frac{4}{45}\frac{P'^1_3(\lambda_{10})}{Q'^1_3(\lambda_{10})}\left[\frac{\lambda_{20}k_2u_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{20})}\frac{k_1^3k_2^2}{s^5}\right.$$

$$\left.+\frac{q_2k_2^2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{20})}\frac{k_1^3k_2^3}{s^6}\right]$$

$$B_4=-\frac{4}{105}\frac{P'^1_4(\lambda_{10})\lambda_{20}k_2u_2}{Q'^1_4(\lambda_{10})(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{20})}\frac{k_1^4k_2^2}{s^6}$$

$$B_p=0,\qquad (p=5,6\ldots ad\ inf.)$$

From (35) to (38), we have in a similar way,

$$b_1=-\frac{\lambda_{20}k_2u_2}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{20})}\left(1+\frac{4}{9}\frac{P'^1_1(\lambda_{10})P'^1_1(\lambda_{20})}{Q'^1_1(\lambda_{10})Q'^1_1(\lambda_{20})}\frac{k_1^3k_2^3}{s^6}\right)$$

$$-\frac{2}{3}\frac{P'^1_1(\lambda_{20})\lambda_{10}k_1u_1}{Q'^1_1(\lambda_{20})Q'^1_1(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}}\frac{k_2k_1^2}{s^2}\left(1+\frac{6}{5}\frac{k_1^2+k_2^2}{s^2}\right)$$

$$-\frac{2}{5}\frac{P'^1_1(\lambda_{20})k_1k_2q_1}{Q'^1_1(\lambda_{20})Q'^1_2(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}}\frac{k_1^4}{s^4}\left\{1+\frac{10}{s^2}\left(\frac{k_1^2}{7}+\frac{k_2^2}{5}\right)\right\}$$

$$b_2=-\frac{k_2^2q_2}{3(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{20})}-\frac{2}{9}\frac{P'^1_2(\lambda_{20})\lambda_{10}k_1u_1}{Q'^1_2(\lambda_{20})Q'^1_1(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}}\frac{k_1^2k_2^2}{s^4}$$

$$\times\left\{1+\frac{10}{s^2}\left(\frac{k_1^2}{5}+\frac{k_2^2}{7}\right)\right\}-\frac{8}{45}\frac{P'^1_2(\lambda_{20})k_1^2q_1}{Q'^1_2(\lambda_{20})Q'^1_2(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}}\frac{k_1^3k_2^2}{s^5}$$

$$b_3=-\frac{4}{45}\frac{P'^1_3(\lambda_{20})}{Q'^1_3(\lambda_{20})}\left[\frac{\lambda_{10}k_1u_1}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{10})}\frac{k_1^2k_2^3}{s^5}\right.$$

$$\left.+\frac{q_1k_1^2}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{10})}\frac{k_1^3k_2^3}{s^6}\right]$$

$$b_4=-\frac{4}{105}\frac{P'^1_4(\lambda_{20})}{Q'^1_4(\lambda_{20})}\cdot\frac{\lambda_{10}k_1u_1}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{10})}\frac{k_1^2k_2^4}{s^6}$$

$$b_p=0,\qquad (p=5,6,\ldots ad\ inf.)$$

From (39) to (44), we have,

$$C_1=-\frac{\lambda_{10}k_1v_1}{(\lambda_{10}^2-1)^{\frac{1}{2}}Q_1^{1}(\lambda_{10})}\left(1+\frac{4}{9}\,\frac{P'^1_1(\lambda_{10})P'^1_1(\lambda_{20})}{Q'^1_1(\lambda_{10})Q'^1_1(\lambda_{20})}\,\frac{k_1^3k_2^3}{s^6}\right)$$

$$-\frac{2}{3}\,\frac{P'^1_1(\lambda_{10})\lambda_{20}k_2v_2}{Q'^1_1(\lambda_{10})Q'^1_1(\lambda_{20})(\lambda_{20}^2-1)^{\frac{1}{2}}}\,\frac{k_1k_2^3}{s^3}\left(1+\frac{6}{5}\,\frac{k_1^2k_2^2}{s^2}\right)$$

$$+\frac{2}{5}\,\frac{P'^1_1(\lambda_{10})k_1k_2p_2}{Q'^1_1(\lambda_{10})Q'^1_2(\lambda_{20})(\lambda_{20}^2-1)^{\frac{1}{2}}}\,\frac{k_1^2k_2^4}{s^4}\left\{1+\frac{10}{s^2}\left(\frac{k_1^2}{5}+\frac{k_2^2}{7}\right)\right\}$$

$$C_2=\frac{k_1^2p_1}{3(\lambda_{10}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{10})}-\frac{2}{9}\,\frac{P'^1_2(\lambda_{10})\lambda_{20}k_2v_2}{Q'^1_2(\lambda_{10})Q'^1_1(\lambda_{20})(\lambda_{20}^2-1)^{\frac{1}{2}}}$$

$$\times\frac{k_1^2k_2^3}{s^4}\left\{1+\frac{10}{s^2}\left(\frac{k_1^2}{7}+\frac{k_2^2}{5}\right)\right\}$$

$$+\frac{8}{45}\,\frac{P'^1_2(\lambda_{10})k_2^2p_2}{Q'^1_2(\lambda_{10})Q'^1_2(\lambda_{20})(\lambda_{20}^2-1)^{\frac{1}{2}}}\,\frac{k_1^2k_2^3}{s^5}$$

$$C_3=-\frac{4}{45}\,\frac{P'^1_3(\lambda_{10})}{Q'^1_3(\lambda_{10})}\left[\frac{\lambda_{20}k_2v_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{20})}\,\frac{k_1^3k_2^3}{s^6}\right.$$

$$\left.-\frac{p_2k_2^2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{20})}\,\frac{k_1^3k_2^3}{s^6}\right]$$

$$C_4=-\frac{4}{105}\,\frac{P'^1_4(\lambda_{10})\lambda_{20}k_2v_2}{Q'^1_4(\lambda_{10})(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{20})}\,\frac{k_1^4k_2^3}{s^6}$$

$$C_p=0,\qquad (p=5,6,\ldots\textit{ad inf.})$$

$$c_1=-\frac{\lambda_{20}k_2v_2}{(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_1(\lambda_{20})}\left(1+\frac{4}{9}\,\frac{P'^1_1(\lambda_{10})P'^1_1(\lambda_{20})}{Q'^1_1(\lambda_{10})Q'^1_1(\lambda_{20})}\,\frac{k_1^3k_2^3}{s^6}\right)$$

$$\frac{2}{3}\,\frac{P'^1_1(\lambda_{20})\lambda_{10}k_1v_1}{Q'^1_1(\lambda_{20})Q'^1_1(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}}\,\frac{k_2k_1^3}{s^3}\left(1+\frac{6}{5}\,\frac{k_1^2+k_2^2}{s^2}\right)$$

$$+\frac{2}{5}\,\frac{P'^1_1(\lambda_{20})k_1k_2p_1}{Q'^1_1(\lambda_{20})Q'^1_2(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}}\,\frac{k_1^4}{s^4}\left\{1+\frac{10}{s^2}\left(\frac{k_1^2}{7}+\frac{k_2^2}{5}\right)\right\}$$

$$c_2=\frac{k_2^2p_2}{3(\lambda_{20}^2-1)^{\frac{1}{2}}Q'^1_2(\lambda_{20})}$$

$$-\frac{2}{9}\,\frac{P'^1_2(\lambda_{20})\lambda_{10}k_1v_1}{Q'^1_2(\lambda_{20})Q'^1_1(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}}\,\frac{k_1^3k_2^2}{s^4}\left\{1+\frac{10}{s^2}\left(\frac{k_1^2}{5}+\frac{k_2^2}{7}\right)\right\}$$

$$+\frac{8}{45}\,\frac{P'^1_2(\lambda_{20})k_1^2p_1}{Q'^1_2(\lambda_{20})Q'^1_2(\lambda_{10})(\lambda_{10}^2-1)^{\frac{1}{2}}}\,\frac{k_1^3k_2^2}{s^6}$$

$$c_3 = -\frac{4}{45}\,\frac{P'^1_3(\lambda_{20})}{Q'^1_3(\lambda_{20})}\left[\frac{\lambda_{10}k_1v_1}{(\lambda^2_{10}-1)^{\frac{1}{2}}Q'^1_1(\lambda_{10})}\,\frac{k_1^2k_2^3}{s^5}\right.$$

$$\left. - \frac{p_1k_1^3}{(\lambda^2_{10}-1)^{\frac{1}{2}}Q'^1_2(\lambda_{10})}\,\frac{k_1^3k_2^3}{s^6}\right]$$

$$c_4 = -\frac{4}{105}\,\frac{P'^1_4(\lambda_{20})}{Q'^1_4(\lambda_{20})}\,\frac{\lambda_{10}k_1v_1}{(\lambda^2_{10}-1)^{\frac{1}{2}}Q'^1_1(\lambda_{10})}\,\frac{k_1^2k_2^4}{s^6}$$

$c_p = 0, \qquad (p = 5, 6, \ldots$ *ad inf.*)

$\therefore$ From (5),

$$\phi = \sum_{n=1}^{4}\{A_nP_n(\mu_1)Q_n(\lambda_1) + a_nP_n(\mu_2)Q_n(\lambda_2)$$

$$+ P^1_n(\mu_1)Q^1_n(\lambda_1)(B_n\cos\omega + C_n\sin\omega)$$

$$+ P^1_n(\mu_2)Q^1_n(\lambda_2)(b_n\cos\omega + c_n\sin\omega)\}$$

where the constants are determined above. ... (50)

In a similar way the expression for ϕ may be determined correct to any power of $\left(\frac{\text{linear dimension}}{\text{central distance}}\right)$.

6. ϕ *in spherical harmonics.*

ϕ may also be expressed in spherical harmonics by the help of the following theorem :

$$P^\sigma_n(\mu)Q^\sigma_n(\lambda) = (-)^\sigma\,\frac{2^n\lfloor n\,\lfloor n+\sigma}{\lfloor 2n+1}\,k^{n+1}\left[\frac{P^\sigma_n(\cos\theta)}{r^{n+1}}\right.$$

$$\left. + \frac{\lfloor n+2-\sigma}{\lfloor n-\sigma}\,\frac{k_2^2}{2(2n+3)}\,\frac{P^\sigma_{n+2}(\cos\theta)}{r^{n+3}} + \text{etc.}\right]$$

for all integral values of σ including zero. ... (51)

Also, $\qquad \frac{1}{\lambda_{10}} = e_1, \quad \frac{1}{\lambda_{20}} = e_2, \quad k_1 = a_1e_1, \; k_2 = a_2e_2.$

7. *Motion of two oblate spheroids.*

If in Art. 2, we write $\frac{k'_1}{i}$ for k and $i\lambda$ for λ,

we have,
$$x_1=k'_1(1-\mu_1^2)^{\frac{1}{2}}(\lambda_1^2+1)^{\frac{1}{2}}\cos\omega$$

$$y_1=k'_1(1-\mu_1^2)^{\frac{1}{2}}(\lambda_1^2+1)^{\frac{1}{2}}\sin\omega$$

$$z_1=k'_1\mu_1\lambda_1$$

$$x_2=k'_2(1-\mu_2^2)^{\frac{1}{2}}(\lambda_2^2+1)^{\frac{1}{2}}\cos\omega$$

$$y_2=k'_2(1-\mu_2^2)^{\frac{1}{2}}(\lambda_2^2+1)^{\frac{1}{2}}\sin\omega$$

$$z_2=k'_2\mu_2\lambda_2$$

where (λ_1,μ_1,ω), (λ_2,μ_2,ω) are two systems of planetary spheroidal co-ordinates so that $\lambda_1=\lambda_{10}$ and $\lambda_2=\lambda_{20}$ on the given spheroids. Hence, by writing $\frac{k'}{i}$ for k and $i\lambda$ for λ in (5) and the expressions for the constants in the case of prolate spheroids, the corresponding results for oblate spheroids are obtained at once.

8. If $u_1, v_1, u_2, v_2, p_1, q_1, p_2, q_2$ are all zero, evidently the problem reduces to that of the motion[1] of two spheroids in an infinite liquid along their common axis of revolution. In this case, we have from 5(A),

$$A_1=-\frac{k_1w_1}{Q'_1(\lambda_{10})},\quad a_1=-\frac{k_2w_2}{Q'_1(\lambda_{20})}$$ and all B's, b's, C's, c's zero

and from 5(B),

$$A_1=-\frac{k_1w_1}{Q'(\lambda_{10})}+\frac{2}{3}\,\frac{k_2w_2}{Q'_1(\lambda_{10})Q'(\lambda_{20})}\,\frac{k_1k_2^2}{s^3}$$

$$a_1=-\frac{k_2w_2}{Q'_1(\lambda_{20})}+\frac{2}{3}\,\frac{k_1w_1}{Q'_1(\lambda_{10})Q'(\lambda_{20})}\,\frac{k_2k_1^2}{s^3}.$$

All B's, C's, b's, c's are zero.

Both these results have already been obtained by Dr. Datta.[2]

[1] Dr. Bibhutibhushan Datta.—"Americ. Journ. Math." *ibid.*

[2] "Americ. Journ. Math." *ibid.*, p. 141.

On the evaluation of some factorable continuants

BY

Satischandra Chakrabarti, M.Sc.

There is a special class of continuants which are resolvable into linear factors, the earliest continuants of this class being those of Sylvester[1] and Painvin.[2] Subsequently Dr. T. Muir[3] and W. H. Metzler[4] took up the subject in right earnest and obtained some continuants of this class. Haripada Datta in his paper, "On the Failure of Heilermann's Theorem"[5] pointed out that "for every finite series we obtain a factorizable continuant." In the same paper of Mr. Datta, a continuant derived from recurring series has been given Here in this paper in article 7, we have evaluated this continuant determinantally. There is another factorable continuant obtained from $\left(\frac{x+1}{x-1}\right)^q$ which is also treated in articles 5 and 6.[6] In evaluating the second continuant we are to apply some algebraic relations which are first established.

[1] Sylvester J. J. "Theoreme sur les determinants de M. Sylvester." Nouv Annals de Math., xiii, p. 305 ; or The Theory of Determinants in the Historical Order of Development by Muir T., Vol. 2, p. 425.

[2] Painvin L. "Sur un certain systeme d'equations lineaires." Journ. (de Liouville) de Math (2) iii, pp. 41-46 ; or The Theory of Determinants in the Historical Order of Development by Muir T., Vol. 2, pp. 432-434.

[3] Muir T. "Continuants resolvable into linear factors" Trans. Edin. Roy. Soc. 41, 1905 (343-358). Muir T. "Factorizable continuants" Trans. S. Afric Philos. Soc. 15 pt. 1, 1904 (29-33).

[4] W. H. Metzler. "Some factorable continuants," Edin. Proc. Roy. Soc., 34, 1914 (223-229).

[5] Proc. Edin Math. Soc., Vol. 35 (part 2), session 1916-1917 ; or University Edin. Math. Depart. session 1917, Research paper No. 7, p. 12.

[6] Haripada Datta "On the Theory of continued fractions." Proc Edin Math. Soc., Vol. 34 (part 2), session 1915-1916 ; or University Edin. Math. Dept. session 1916, Research paper No. 4, p. 9, Ex. 2.

$$
\begin{aligned}
1.\quad & \{(x+1)(x+3)(x+5)\ldots(x+2r-1)\}\\
&=\{(x-\delta)(x-\delta-2)(x-\delta-4)\ldots(x-\delta-\overline{2r-2})\}\\
&+(\delta+2r-1)\,{}^{r}C_1\{(x-\delta)(x-\delta-2)\ldots(x-\delta-\overline{2r-4})\}\\
&+(\delta+2r-1)(\delta+2r-3)\,{}^{r}C_2\{(x-\delta)(x-\delta-2)\ldots(x-\delta-\overline{2r-6})\}\\
&+\ldots+\{(\delta+2r-1)(\delta+2r-3)\ldots(\delta+5)(\delta+3)\}\,{}^{r}C_{r-1}(x-\delta)\\
&+\{(\delta+2r-1)(\delta+2r-3)\ldots(\delta+3)(\delta+1)\}\ \text{identically} \qquad \ldots \quad (1)
\end{aligned}
$$

Let us take the particular case,

$$
\begin{aligned}
&(x+1)(x+3)(x+5)(x+7)=(x-\delta)(x-\delta-2)(x-\delta-4)(x-\delta-6)\\
&+(\delta+7)\,{}^{4}C_1(x-\delta)(x-\delta-2)(x-\delta-4)\\
&+(\delta+7)(\delta+5)\,{}^{4}C_2(x-\delta)(x-\delta-2)\\
&+(\delta+7)(\delta+5)(\delta+3)\,{}^{4}C_3(x-\delta)+(\delta+7)(\delta+5)(\delta+3)(\delta+1) \quad \ldots \quad (2)
\end{aligned}
$$

and let α and β respectively denote the left-hand-side and the right-hand-side expression of (2).

When $x=\delta$, $\alpha=\beta=(\delta+1)(\delta+3)(\delta+5)(\delta+7)$ and when $x=-1$, $\alpha=0, \beta=(-1)^4(\delta+1)(\delta+3)(\delta+5)(\delta+7)\{1-{}^4C_1+{}^4C_2-{}^4C_3+{}^4C_4\}=0$. When $x=-3$, -5 or -7, we can show, by means of difference formulae that in each of these substitutions $\alpha=\beta=0$. Thus the equation (2) is satisfied for more than four values of x; hence it is an identity. The general case may be similarly proved.

$$
\begin{aligned}
2.\quad & \{(a+2r-1)(a+2r+1)\ldots(a+4r-3)\}\\
&=\{a(a+2)(a+4)\ldots(a+2r-2)\}\,{}^{2r}C_{2r}\\
&+1\cdot{}^{2r}C_{2r-2}\{(a+2)(a+4)\ldots(a+2r-2)\}\\
&+1\cdot 3\cdot{}^{2r}C_{2r-4}\{(a+4)(a+6)\ldots(a+2r-2)\}+\ldots\\
&+\{1\cdot3\cdot5\ldots(2r-2k-3)\}\,{}^{2r}C_{2k+2}\{(a+2r-2k-2)(a+2r-2k)\ldots\\
&\qquad (a+2r-2)\}+\ldots\{1\cdot3\ldots(2r-3)\}\,{}^{2r}C_2\,(a+2r+2)\\
&\qquad\qquad +\{1\cdot3\cdot5\ldots(2r-1)\}\,{}^{2r}C_0\ \text{identically} \qquad \ldots \quad (3)
\end{aligned}
$$

Suppose

$$
\begin{aligned}
&\{(a+2r-1)(a+2r+1)\ldots(a+4r-3)\}=\{a(a+2)\ldots(a+2r-2)\}\\
&+A_2\{(a+2)(a+4)\ldots(a+2r-2)\}+A_3\{(a+4)(a+6)\ldots(a+2r-2)\}\\
&+\ldots+A_{r-k-1}\{(a+2r-2k-4)(a+2r-2k-2)\ldots(a+2r-2)\}\\
&+A_{r-k}\{(a+2r-2k-2)(a+2r-2k)\ldots(a+2r-2)\}+\ldots\\
&\qquad +A_r(a+2r-2)+A_{r+1} \qquad \ldots \quad (4)
\end{aligned}
$$

where $A_2, A_3, \ldots A_{r+1}$ which are independent of a, are to be determined.

In (4) put $a=-2r+2$, then $A_{r+1}=\{1\cdot3\cdot5\ldots(2r-1)\}^{2}\,{}^{r}C_{0}$

$a=-2r+4$, $\quad A_{r}=\{1\cdot3\cdot5\ldots(2r-3)\}^{2}\,{}^{r}C_{2}$

$a=-2r+6$, $\quad A_{r-1}=\{1\cdot3\ldots(2r-5)\}^{2}\,{}^{r}C_{4}$

Thus we see that there is a formula for A_{r+1}, A_r and A_{r-1}. Let us assume that this formula is true for all cases from A_{r+1} to A_{r-k}. It is required to show that the formula also holds in the case of A_{r-k-1}.

In (4) putting $a=-2r+2k+6$, we have,

$$\{2\cdot4\cdot6\cdots(2k+4)\}A_{r-k-1}=\{(2k+5)(2k+7)\ldots(2r+2k+3)\}$$
$$-\{4\cdot6\ldots(2k+4)\}A_{r-k}-\{6\cdot8\ldots(2k+4)\}A_{r-k+1}-\cdots$$
$$-\{(2k-2m+6)(2k-2m+8)\ldots(2k+4)\}A_{r-m+1}-\ldots$$
$$-\{(2k+2)(2k+4)\}A_{r-1}-(2k+4)A_{r}-A_{r+1}$$

$$=\{(2k+5)(2k+7)\ldots(2r-1)\}\Big[\ \{(2r+1)(2r+3)\ldots(2r+2k+3)\}$$
$$-\frac{\{4\cdot6\ldots(2k+4)\}\{1\cdot3\ldots(2k+3)\}}{\{(2r-1)(2r-3)\ldots(2r-2k-1)\}}\ \frac{\underline{|2r}}{\underline{|2r-2k-2}\ \underline{|2k+2}}$$
$$-\frac{\{6\cdot8\ldots(2k+4)\}\{1\cdot3\ldots(2k+3)\}}{\{(2r-1)(2r-3)\ldots(2r-2k+1)\}}\ \frac{\underline{|2r}}{\underline{|2r-2k}\ \underline{|2k}}-\cdots$$
$$-\frac{\{(2k-2m+6)(2k-2m+8)\ldots(2k+4)\}\{1\cdot3\ldots(2k+3)\}\underline{|2r}}{\{(2r-1)(2r-3)\ldots(2r-2m+1)\}\ \underline{|2r-2m}\ \underline{|2m}}-\cdots$$
$$-\frac{(2k+4)\{1\cdot3\ldots(2k+3)\}}{2r-1}\ \frac{\underline{|2r}}{\underline{|2r-2}\ \underline{|2}}\ -\{1\cdot3\ldots(2k+3)\}\ \Big]$$

$$=\{(2k+5)(2k+7)\ldots(2r-1)\}\big[\{(2r+1)(2r+3)\ldots(2r+2k+3)\}$$
$$-(2k+3)\,{}^{k+2}C_{1}\{2r(2r-2)\ldots(2r-2k)\}$$
$$-(2k+3)(2k+1)\,{}^{k+2}C_{2}\{2r(2r-2)\ldots(2r-2k+2)\}-\ldots$$
$$-\{(2k+3)(2k+1)\ldots(2m+1)\}\,{}^{k+2}C_{k-m+2}\{2r(2r-2)\ldots(2r-2m+2)\}$$
$$-\ldots-\{(2k+3)(2k+1)\ldots5\cdot3\}\,{}^{k+2}C_{k+1}2r$$
$$-\{(2k+3)(2k+1)\ldots3\cdot1\}\,{}^{k+2}C_{k+2}\big]$$

For $\dfrac{\{(2k-2m+6)(2k-2m+8)\dots(2k+4)\}\{1\cdot3\dots(2k+3)\}}{\{(2r-1)(2r-3)\dots(2r-2m+1)\}}\dfrac{\lfloor 2r}{\lfloor 2r-2m\,\lfloor 2m}$

$$=\frac{\{(2k-2m+6)(2k-2m+8)\dots(2k+4)\}\{1\cdot3\dots(2k+3)\}\{2r(2r-2)\dots(2r-2m+2)\}}{\lfloor 2m}$$

$$=\frac{\lfloor 2k+4\ \{2r(2r-2)\dots(2r-2m+2)\}}{\{2\cdot4\cdot6\dots(2k-2m+4)\}\lfloor 2m}$$

$$=\{(2k+3)(2k+1)\dots(2m+1)\}\,{}^{k+2}C_{k-m+2}\{2r(2r-2)\dots(2r-2m+2)\}$$

Hence by the identity obtained from (1) by putting $\delta=0$ we have,

$$\{2\cdot4\cdot6\dots(2k+4)\}A_{r-k-1}$$

$$=\{(2k+5)(2k+7)\dots(2r-1)\}\{2r(2r-2)(2r-4)\dots(2r-2k-2)\}$$

$$=\frac{\lfloor 2r}{\lfloor 2k+4\{(2k+6)(2k+8)\cdots(2r-2k-4)\}}$$

$$\therefore\ A_{r-k-1}=\{1\cdot3\cdot5\dots(2r-2k-5)\}\,{}^{2r}C_{2k+4}.$$

Thus the formula holds universally. Hence the identity is established,

3. $\{(a+2r-1)(a+2r+1)(a+2r+3)\dots(a+4r-5)\}$

$$=\{a(a+2)\dots(a+2r-4)\}+1\cdot{}^{2r-1}C_{2r-3}\{(a+2)(a+4)\dots(a+2r-4)\}$$

$$+1\cdot3\cdot{}^{2r-1}C_{2r-5}\{(a+4)(a+6)\dots(a+2r-4)\}$$

$$+1\cdot3\cdot5\cdot{}^{2r-1}C_{2r-7}\{(a+6)(a+8)\dots(a+2r-4)\}+\cdots$$

$$+\{1\cdot3\cdot5\dots(2r-5)\}\,{}^{2r-1}C_3(a+2r-4)$$

$$+\{1\cdot3\cdot5\dots(2r-3)\}\,{}^{2r-1}C_1 \text{ identically} \qquad \dots \quad (5)$$

This theorem is established by proceeding in the same manner as the theorem (3), the identity to be applied being obtained by putting $\delta=2$ in the theorem (1).

4. Here in this article we shall give another identity,

$$\frac{1}{\lfloor n}-\frac{1}{\lfloor n-1\,\lfloor 1}+\frac{1}{\lfloor n-2\,\lfloor 2}\,\frac{a+1}{a}-\frac{1}{\lfloor n-3\,\lfloor 3}\,\frac{a+3}{a}$$

$$+\frac{1}{\lfloor n-4\,\lfloor 4}\,\frac{(a+3)(a+5)}{a(a+2)}-\frac{1}{\lfloor n-5\,\lfloor 5}\,\frac{(a+5)(a+7)}{a(a+2)}$$

$$+\frac{1}{\lfloor n-6\,\lfloor 6}\,\frac{(a+5)(a+7)(a+9)}{a(a+2)(a+4)}-\frac{1}{\lfloor n-7\,\lfloor 7}\,\frac{(a+7)(a+9)(a+11)}{a(a+2)(a+4)}+$$

$$\dots=0 \text{ or } \frac{\{1\cdot3\cdot5\dots(n-1)\}}{\lfloor n\ a(a+2)\dots(a+n-2)}$$

according as n is odd or even ; the last term of the series is,

$$\frac{\{(a+n)(a+n+2)\dots(a+2n-3)\}}{\underline{|n}\{a(a+2)(a+4)\dots(a+n-3)\}}$$

or
$$\frac{\{(a+n-1)(a+n+1)\dots(a+2n-3)\}}{\underline{|n}\{a(a+2)(a+4)\dots(a+n-2)\}} \qquad \dots \quad (6)$$

according as n is odd or even.

Let $n=9$ an odd number, then the series becomes

$$\frac{1}{\underline{|9}}-\frac{1}{\underline{|8}\,\underline{|1}}+\frac{1}{\underline{|7}\,\underline{|2}}\,\frac{a+1}{a}-\frac{1}{\underline{|6}\,\underline{|3}}\,\frac{a+3}{a}+\frac{1}{\underline{|5}\,\underline{|4}}\,\frac{(a+3)(a+5)}{a(a+2)}-\frac{1}{\underline{|4}\,\underline{|5}}\,\frac{(a+5)(a+7)}{a(a+2)}$$

$$+\frac{1}{\underline{|3}\,\underline{|6}}\,\frac{(a+5)(a+7)(a+9)}{a(a+2)(a+4)}-\frac{1}{\underline{|2}\,\underline{|7}}\,\frac{(a+7)(a+9)(a+11)}{a(a+2)(a+4)}$$

$$+\frac{1}{\underline{|1}\,\underline{|8}}\,\frac{(a+7)(a+9)(a+11)(a+13)}{a(a+2)(a+4)(a+6)}$$

$$-\frac{1}{\underline{|9}}\,\frac{(a+9)(a+11)(a+13)(a+15)}{a(a+2)(a+4)(a+6)} \qquad \dots \quad (7)$$

Let $\quad {}^na_r={}^nC_{r-1}-{}^rC_1\,{}^nC_r+{}^{r+1}C_2\,{}^nC_{r+1}-{}^{r+2}C_3\,{}^nC_{r+2}$

$$+\dots(-1)^{n-r+1}\;{}^nC_{n-r+1}\,{}^nC_n.$$

Then

$${}^na_r={}^nC_{r-1}\{1-{}^{n-r+1}C_1+{}^{n-r+1}C_2-\dots+(-1)^{n-r+1}\;{}^{n-r+1}C_{n-r+1}\}=0$$

for $\quad {}^qC_k\cdot{}^mC_q={}^mC_{q-k}\cdot{}^{m-q+k}C_k.$

Hence $\quad {}^na_1={}^na_2={}^na_3=\dots={}^na_n=0$ but ${}^na_{n+1}={}^nC_n=1. \qquad \dots \quad (8)$

Now if $u_1=a(a+2)(a+4)(a+6)$ which is the denominator of the last term of the series (7),

$$u_2=(a+2)(a+4)(a+6),\ u_3=(a+4)(a+6) \text{ and } u_4=a+6.$$

then the numerator of the series (7) which is

$a(a+2)(a+4)(a+6)-{}^9C_1a(a+2)(a+4)(a+6)$

$+{}^9C_2(a+1)(a+2)(a+4)(a+6)-{}^9C_3(a+3)(a+2)(a+4)(a+6)$

$+{}^9C_4(a+3)(a+5)(a+4)(a+6)-\dots-{}^9C_9(a+9)(a+11)(a+13)(a+15)$

may be written as

$$u_1 - {}^9C_1u_1 + {}^9C_2(u_1+u_2) - {}^9C_3(u_1 + {}^3C_1u_2) + {}^9C_4(u_1 + {}^4C_2u_2$$
$$+1\cdot 3\cdot {}^4C_0u_3) - \ldots + {}^9C_8(u_1 + {}^8C_6u_2 + 1\cdot3\cdot{}^8C_4u_3 + 1\cdot3\cdot5\cdot{}^8C_2u_4$$
$$+1\cdot3\cdot5\cdot7) - {}^9C_9(u_1 + {}^9C_7u_2 + 1\cdot3\cdot{}^9C_5u_3 + 1\cdot3\cdot5\cdot{}^9C_3u_4$$
$$+1\cdot3\cdot5\cdot7\cdot{}^9C_1)$$

by applying the theorem (3) to odd terms and the theorem (5) to even terms.

Hence the numerator

$$=u_1{}^9a_1 + u_2{}^9a_3 + 1\cdot3u_3{}^9a_5 + 1\cdot3\cdot5u_4{}^9a_7 + 1\cdot3\cdot5\cdot7{}^9a_9 = 0 \text{ by (8).}$$

Thus the series vanishes.

When $n=6$ an even number and if.

$$u_1 = a(a+2)(a+4),\ u_2 = (a+2)(a+4) \text{ and } u_3 = a+4,$$

the numerator becomes,

$$u_1 - {}^6C_1u_1 + {}^6C_2(u_1+u_2) - {}^6C_3(u_1 + {}^3C_1u_2) + {}^6C_4(u_1 + {}^4C_2u_2 + 1\cdot3u_3)$$
$$-{}^6C_5(u_1 + {}^5C_3u_2 + 1\cdot3\cdot{}^5C_1u_3) + {}^6C_6(u_1 + {}^6C_4u_2 + 1\cdot3\cdot{}^6C_2u_3 + 1\cdot3\cdot5)$$
$$= {}^6a_1u_1 + {}^6a_3u_2 + 1\cdot3{}^6a_5u_3 + 1\cdot3\cdot5 = 1\cdot3\cdot5.$$

Hence when $n=6$, the series $= \dfrac{1\cdot3\cdot5}{\underline{|6}\, a(a+2)(a+4)}$.

The general case may be treated exactly in the same manner.

5. We shall now evaluate the factorable continuant

$$D_5 = \begin{vmatrix} x-5, & -1 & & & \\ 5^2-1^2 & 3x & -1 & & \\ & 5^2-2^2 & 5x & -1 & \\ & & 5^2-3^2 & 7x & -1 \\ & & & 5^2-4^2 & 9x \end{vmatrix}$$

On this if we proform the operation

$(-1)^{5-1}\,\pi\,(5-1, 1)\ \text{col}_5 + (-1)^{5-2}\,\pi\,(5-1, 2)\ \text{col}_4$

$+(-1)^{5-3}\,\pi\,(5-1, 3)\ \text{col}_3 + (-1)^{5-4}\,\pi\,(5-1, 4)\ \text{col}_2$

$+(-1)^{5-5}\,\pi\,(5-1, 5)\ \text{col}_1$ where $\pi\,(n, r)$ denotes the product $\{n(n-1)\,(n-2)\ldots r\}$, $n-r$ being a positive integer or zero and the product being taken as unity when r is greater than n

then we have $D_5 = \dfrac{9(x-1)}{24}$ $\begin{vmatrix} x-5 & -1 & & 0 & 1 \\ 24, & 3x & -1 & 0 & -12 \\ & 21 & & -1 & 60 \\ & & 16 & & -168 \\ & & & 1 & 24 \end{vmatrix}$

Then by the operation

$2 \times \mathrm{col}_5 + (-1)^{5-2}\, \pi\, (5-2,\ 1)\ (2\cdot 5-2)\ \mathrm{col}_4$

$\quad + (-1)^{5-3}\, \pi\, (5-2,\ 2)\ (2\cdot 5-3)\ \mathrm{col}_3$

$\quad + (-1)^{5-4}\, \pi\, (5-2,\ 3)\ (2\cdot 5-4)\ \mathrm{col}_2$

$\quad + (-1)^{5-5}\, \pi\, (5-2,\ 4)\ (2\cdot 5-5)\ \mathrm{col}_1$, we obtain

$$D_5 = -3\,(x-1)^2 \begin{vmatrix} x-5, & -1, & 0, & 5 \\ 24, & 3x, & -1, & -54 \\ & 21, & 5x & 210 \\ & & 1 & -21 \end{vmatrix}$$

Now performing the operation

$2 \times 2\ \mathrm{col}_4 + (-1)^{5-3}\, \pi\, (5-3,\ 1)\ \pi\, (2\cdot 5-3,\ 2\cdot 5-4)\ \mathrm{col}_3$

$\quad + (-1)^{5-4}\, \pi\, (5-3,\ 2)\ \pi\, (2\cdot 5-4,\ 2\cdot 5-5)\ \mathrm{col}_2$

$\quad + (-1)^{5-5}\, \pi\, (5-3,\ 3)\ \pi\, (2\cdot 5-5,\ 2\cdot 5-6)\ \mathrm{col}_1$, we have

$$D_5 = \frac{21 \times 3\,(x-1)}{4} \begin{vmatrix} x-5, & -1, & 20 \\ 24, & 3x, & -180 \\ & 1 & 20 \end{vmatrix}$$

On this last determinant perform the operation

$2 \times 3\ \mathrm{col}_3 + (-1)^{5-4}\, \pi\, (5-4,\ 1)\ \pi\, (2\cdot 5-4,\ 2\cdot 5-6)\ \mathrm{col}_2$

$\quad + (-1)^{5-5}\, \pi\, (5-4,\ 2)\ \pi\, (2\cdot 5-5,\ 2\cdot 5-7)\ \mathrm{col}_1$ then we obtain

$$D_5 = -7 \times 9\,(x-1)^4 \begin{vmatrix} x-5, & 60 \\ 1 & -15 \end{vmatrix}$$

On this again perform the operation

$2 \times 4\ \mathrm{col}_2 + (-1)^{5-5}\, \pi\, (5-5,\ 1)\ \pi\, (2\cdot 5-5,\ 2\cdot 5-8)\ \mathrm{col}_1$

Then we have $D_5 = \dfrac{-7\times 9\,(x-1)^5}{8} \begin{vmatrix} x-5, & 120 \\ 1 & 0 \end{vmatrix} = 1\times 3\times 5\times 7 \times 9\,(x-1)^5$

Similarly we can show than $D_n = 1\cdot 3\cdot 5 \ldots (2n-1)\,(x-1)^n$.

6. In the case of D_n, if no factor is removed from the last column of the determinant that results from any operation, we shall have to perform the following operations:—

$$(-1)^{n-1}\,\pi\,(n-1,\,1)\,\mathrm{col}_n + (-1)^{n-2}\,\pi\,(n-1,\,2)\,\mathrm{col}_{n-1}$$
$$+(-1)^{n-3}\,\pi\,(n-1,\,3)\,\mathrm{col}_{n-2} + \ldots + (n-1)\,(n-2)\,\mathrm{col}_3$$
$$-(n-1)\,\mathrm{col}_2 + \mathrm{col}_1 = \mathrm{col}_n^{(1)} \qquad \ldots \quad (a_1)$$

$$2\,\mathrm{col}_n^{(1)} + (-1)^{n-2}\,\pi\,(n-2,\,1)\,(2n-2)\,(x-1)\,\mathrm{col}_{n-1}$$
$$+(-1)^{n-3}\,\pi\,(n-2,\,2)\,(2n-3)\,(x-1)\,\mathrm{col}_{n-2} + \ldots$$
$$+(n-2)\,(n-3)\,(n+2)\,(x-1)\,\mathrm{col}_3$$
$$-(n-2)\,(n+1)\,(x-1)\,\mathrm{col}_2 + n\,(x-1)\,\mathrm{col}_1 = \mathrm{col}_n^{(2)} \ldots \quad (a_2)$$

$$2\times 2\,\mathrm{col}_n^{(2)} + 0\,\mathrm{col}_{n-1} + (-1)^{n-3}\,\pi\,(n-3,\,1)\,\pi\,(2n-3,\,2n-4)$$
$$\times (x-1)^2\,\mathrm{col}_{n-2} + (-1)^{n-4}\,\pi\,(n-3,\,2)\,\pi\,(2n-4,$$
$$2n-5)\,(x-1)^2\,\mathrm{col}_{n-3} + \ldots + n\,(n-1)\,(x-1)^2\,\mathrm{col}_1$$
$$= \mathrm{col}_n^{(3)} \qquad \ldots \quad (a_3)$$

... ...

$$2\,(n-1)\,\mathrm{col}_n^{(n-1)} + 0\,\mathrm{col}_{n-1} + 0\,\mathrm{col}_{n-2} + \ldots + 0\,\mathrm{col}_2$$
$$+\{2\times 3\times 4 \ldots (n-1)\,n\}\,(x-1)^n\,\mathrm{col}_1 = \mathrm{col}_n^{(n)} \qquad \ldots \quad (a_n)$$

Since any two operatiens of the type

$$A_1\,\mathrm{col}_3 + A_2\,\mathrm{col}_2 + A_3\,\mathrm{col}_1 = \mathrm{col}_3^{(1)}$$
$$B_1\,\mathrm{col}_3^{(1)} + B_2\,\mathrm{col}_2 + B_3\,\mathrm{col}_1 = \mathrm{col}_3^{(2)}$$

may be substituted by the single operation

$$A_1\,B_1\,\mathrm{col}_3 + (B_1\,A_2 + B_2)\,\mathrm{col}_2 + (B_1\,A_3 + B_3)\,\mathrm{col}_1 = \mathrm{col}_3^{(1)}$$

we may substitute for the operations $(a_1, a_2 \ldots a_n)$ a single operation in which the multiplier of $\mathrm{col}_{r-(r-1)}$ will be

$$(-1)^{n-r}\,2^{n-r}\,\underline{|n-1}\,[2^{r-1}\,\pi\,(n-1,\,r)$$
$$+2^{r-2}\,\pi\,(n-2,\,r-1)\,(2n-r)\,(x-1)$$

$$+2^{r-3}\frac{1}{\lfloor 2}\,\pi\,(n-3,\ r-2)\ \pi\,(2n-r,\ 2n-r-1)\ (x-1)^{2}$$

$$+2^{r-4}\frac{1}{\lfloor 3}\,\pi\,(n-4,\ r-3)\ \pi\,(2n-r,\ 2n-r-2)\ (x-1)^{3}+\ldots$$

$$+2\cdot\frac{1}{\lfloor r-2}\,\pi\,(n-r+1,\ 2)\ \pi\,(2n-r,\ 2n-2r+3)\ (x-1)^{r-2}$$

$$+\frac{1}{\lfloor r-1}\,\pi\,(n-r,\ 1)\ \pi\,(2n-r,\ 2n-2r+2)\ (x-1)^{r-1}\Big]$$

Rejecting the factor $\lfloor n-1$ which is common to all the multipliers and writing in the reverse order we have the multiplier of $\text{col}_{n-(r-1)}$

$$=(-1)^{n-r}\,2^{n-r}\Big[\ \frac{1}{\lfloor r-1}\ \pi\,(n-r,\ 1)\ \pi\,(2n-r,\ 2n+2r+2)\ (x-1)^{r-1}$$

$$+\ \frac{2}{\lfloor r-2}\ \pi\,(n-r+1,\ 2)\ \pi\,(2n-r,\ 2n-2r+3)\ (x-1)^{r-2}$$

$$+\ \frac{4}{\lfloor r-3}\ \pi\,(n-r+2,\ 3)\ \pi\,(2n-r,\ 2n-2r+4)\ (x-1)^{r-3}$$

$$+\ldots+2^{r-2}\,\frac{1}{1}\,\pi\,(n-2,\ r-1)\ (2n-r)\ (x-1)+2^{r-1}\,\pi\,(n-1,\ r)\Big]$$

$$=(-1)^{n-r}\,2^{n-r}\,k\Big[\ \frac{1}{\lfloor r-1}\cdot(x-1)^{r-1}+\frac{1}{\lfloor r-2\,\lfloor 1}\cdot(x-1)^{r-2}$$

$$+\frac{1}{\lfloor r-3\,\lfloor 2}\,\frac{a+1}{a}\,(x-1)^{r-3}+\frac{1}{\lfloor r-4\,\lfloor 3}\,\frac{a+3}{a}\,(x-1)^{r-4}$$

$$+\frac{1}{\lfloor r-5\,\lfloor 4}\,\frac{(a+3)\,(a+5)}{a\,(a+2)}\,(x-1)^{r-5}+\ldots$$

$$+\frac{1}{\lfloor r-2p\,\lfloor 2p-1}\,\frac{\{(a+2p-1)\,(a+2p+1)\ldots(a+4p-5)\}}{\{a\,(a+2)\,(a+4)\ldots(a+2p-4)\}}\,(x-1)^{r-2p}$$

$$+\frac{1}{\lfloor r-2p-1\,\lfloor 2p}\cdot\frac{\{(a+2p-1)\,(a+2p+1)\ldots(a+4p-3)\}}{\{a\,(a+2)\,(a+4)\ldots(a+2p-2)\}}$$

$$\times(x-1)^{r-(2p+1)}+\ldots\Big]$$

where $k=\pi\,(n-r,\ 1)\ \pi\,(2n-r,\ 2n-2r+2)$ and $a=2n-2r+3$.

Hence the multiplier of $\mathrm{col}_{n-(r-1)}$ is

$$(-1)^{n-r}\,2^{n-r}\,k\left[\ \frac{1}{\underline{|r-1}}\,x^{r-1}-\frac{1}{\underline{|r-2}}\left\{\frac{1}{\underline{|1}}-\frac{1}{\underline{|1}}\right\}x^{r-2}\right.$$

$$+\frac{1}{\underline{|r-3}}\left\{\frac{1}{\underline{|2}}-\frac{1}{\underline{|1}\,\underline{|1}}+\frac{1}{\underline{|2}}\,\frac{a+1}{a}\right\}x^{r-3}$$

$$-\frac{1}{\underline{|r-4}}\left\{\frac{1}{\underline{|3}}-\frac{1}{\underline{|2}\,\underline{|1}}+\frac{1}{\underline{|1}\,\underline{|2}}\,\frac{a+1}{a}-\frac{1}{\underline{|3}}\,\frac{a+3}{a}\right\}x^{r-4}+\dots$$

$$-\frac{1}{\underline{|r-2p}}\left\{\frac{1}{\underline{|2p-1}}-\frac{1}{\underline{|2p-2}\,\underline{|1}}+\frac{1}{\underline{|2p-3}\,\underline{|2}}\,\frac{a+1}{a}-\dots\right.$$

$$\left.-\frac{1}{\underline{|2p-1}}\,\frac{(a+2p-1)\,(a+2p+1)\dots(a+4p-5)}{a\,(a+2)\dots(a+2p-4)}\right\}x^{r-2p}$$

$$+\frac{1}{\underline{|r-2p-1}}\left\{\frac{1}{\underline{|2p}}-\frac{1}{\underline{|2p-1}\,\underline{|1}}+\frac{1}{\underline{|2p-2}\,\underline{|2}}\,\frac{a+1}{a}\right.$$

$$\left.-\frac{1}{\underline{|2p-3}\,\underline{|3}}\,\frac{a+3}{a}+\dots+\frac{1}{\underline{|2p}}\,\frac{(a+2p-1)\,(a+2p+1)\dots(a+4p-3)}{a\,(a+2)\dots(a+2p-2)}\right\}$$

$$\left.\times\,x^{r-(2p+1)}-\dots\right].$$

So by theorem (6), the multiplier of the $\mathrm{col}_{n-(r-1)}$

$$=(-1)^{n-r}\,2^{n-r}\,k\left[\ \frac{1}{\underline{|r-1}}\,x^{r-1}+\frac{1}{\underline{|r-3}\,\underline{|2}}\,\frac{1}{a}\,x^{r-3}\right.$$

$$+\frac{1\cdot3}{\underline{|r-5}\,\underline{|4}}\,\frac{1}{a\,(a+2)}\,x^{r-5}+\frac{1\cdot3\cdot5}{\underline{|r-7}\,\underline{|6}}\,\frac{1}{a\,(a+2)\,(a+4)}\,x^{r-7}$$

$$\left.+\dots+\frac{1\cdot3\cdot5\dots(2p-1)}{\underline{|r-(2p+1)}\,\underline{|2p}\ a\ (a+2)\dots(a+2p-2)}\,x^{r-(2p+1)}+\dots\right]\dots(9)$$

(i) To illustrate the application of the formula given in (9), let us consider the continuant of the 7th order (i.e $n=7$) then the multiplier of $\mathrm{col}_7=(-1)^6 2^6\pi(6,1)\ \pi(13,14)=46080$ for $\pi(13,14)=1$ (where $r=1$)

$$\dots\qquad\dots\quad \mathrm{col}_6=(-1)^5\ 2^5\ \pi\ (5,\ 1)\ \pi\ (12,\ 12)\left[\frac{1}{\underline{|1}}\ x\right]$$

$$=-46080\,x \text{ for } \pi\ (12,\ 12)=12, \text{ (where } r=2)$$

$$\dots\qquad\dots\qquad\dots\qquad\dots$$

$$\text{multiplier of } \mathrm{col}_1 = \pi\,(7,\,2)\left[\frac{1}{\lfloor 6}\,x^6 + \frac{1}{\lfloor 4\,\lfloor 2\cdot 3}\,x^4 + \frac{1\cdot 3}{\lfloor 2\,\lfloor 4\cdot 3\cdot 5}\,x^2 + \right.$$

$$\left. + \frac{1\cdot 3\cdot 5}{\lfloor 6\cdot 3\cdot 5\cdot 7}\right], \text{ (where } r=7)$$

$$= 7x^6 + 35x^4 + 21x^2 + 1$$

Hence the operation in this case is

$$46080\,\mathrm{col}_7 - 46080x\,\mathrm{col}_6 + (21120x^2 + 1920)\,\mathrm{col}_5 - (5760x^3 + 1920x)\,\mathrm{col}_4 + (1008x^4 + 864x^2 + 48)\,\mathrm{col}_3 - (112x^5 + 224x^3 + 48x)\,\mathrm{col}_2 + (7x^6 + 35x^4 + 21x^2 + 1)\,\mathrm{col}_1$$

and the continuant D_7, by this operation, becomes

$$\frac{1}{46080}\left|\begin{array}{ccccccc} x-7, & -1 & 0 & & & & 7(x-1)^7 \\ 48 & 3x & -1 & & & & 0 \\ & 45 & 5x & -1 & & & 0 \\ & & 40 & 7x & -1 & & 0 \\ & & & 33 & 9x & -1 & 0 \\ & & & & 24 & 11x & 0 \\ & & & & & 13 & 0 \end{array}\right|$$

$$= (-1)^{7-1}\,\frac{7(x-1)^7 \times 48 \times 45 \times 40 \times 33 \times 24 \times 13}{46080} = 1\times 3\times 5\times 7\times 9 \times 11 \times 13\,(x-1)^7$$

7. The continuant

$$\left|\begin{array}{ccccccc} 1, & x & & & & & \\ a+n-1, & a, & x & & & & \\ & 1-n, & a+1, & x & & & \\ & & a(a+n), & a+2, & x & & \\ & \cdots & \cdots & \cdots & \cdots & & \\ & & & & 1-n, & a+2n-3, & x \\ & & & & & a+n-2, & 1 \end{array}\right|_{2n}$$

$=\{a\,(a+1)\,(a+2)\ldots(a+2n-3)\}\,(1-x)^n$ *i.e.* equal to the product of the principal diagonal terms multiplied by $(1-x)^n$. Here the

elements, except the first and last, of the lower minor diagonal are given by

$$e_{2m}=(a+m-2)\,(a+m+n-2)$$

and
$$e_{2m+1}=m\,(m-n)$$

where e_r denotes the element of this diagonal in the r^{th} row.

Let us first consider the particular case when $n=3$

viz.
$$\begin{vmatrix} 1 & x & & & & \\ a+2, & a, & x & & & \\ & -2, & a+1, & x & & \\ & & a\,(a+3), & a+2, & x & \\ & & & -2 & a+3 & x \\ & & & & a+1 & 1 \end{vmatrix}$$

and on this perform the first operation

$$-2a\,(a+1)\,\mathrm{col}_6+2a\,\mathrm{col}_5+2a\,\mathrm{col}_4-2\,\mathrm{col}_3-\mathrm{col}_2+\mathrm{col}_1$$

This enables us to remove the factor $\dfrac{(1-x)\,(a+1)\,(-2)}{2a\,(a+1)}$ and then subtracting the first column from the last, we can remove another factor a and write the co-factor in the form

$$\begin{vmatrix} 1 & x & 0 & 0 & 0 \\ a+2, & a & x & 0 & -1 \\ & -2 & a+1 & x & -2 \\ & & a\,(a+3) & a+2, & -2 \\ & & & 1 & -(a+1) \end{vmatrix}$$

On this co-factor, performing the 2nd operation

$$\mathrm{col}_5+(a+1)\,\mathrm{col}_4-\mathrm{col}_3-\mathrm{col}_2+\mathrm{col}_1$$

we can remove the factor $(1-x)\,a\,(a+3)$ and get the co-factor

$$\begin{vmatrix} 1 & x & 1 \\ a+2, & a, & 1 \\ & -2 & -(a+1) \end{vmatrix}$$

Then substracting the first column from the last we can remove another factor $(a+1)$ and obtain

$$\begin{vmatrix} 1 & x & 0 \\ a+2, & a, & -1 \\ & -2 & -1 \end{vmatrix}$$

Then performing the 3rd operation $2\,\mathrm{col}_3 - \mathrm{col}_2 + \mathrm{col}_1$, we have

$$\frac{1}{2}\begin{vmatrix} 1 & x & 1-x \\ a+2, & a & 0 \\ & -2 & 0 \end{vmatrix} \quad \text{which is equal to } -(a+2)\,(1-x)$$

Thus the result is $a\,(a+1)\,(a+2)\,(a+3)\,(1-x)^3$.

In the general case, if m_k denotes the multiplier of the k^{th} column and l that of the last column, then

$$\text{in the first operation} \ldots \begin{cases} m_{2r} = (-1)^{\frac{2r\,(2r-1)}{2}}\,\pi(a+r-2,\, a)\,\pi(n-1,\, n-r+1) \\ m_{2r+1} = (-1)^{\frac{(2r+1)\,2r}{2}}\,\pi(a+r-2,\, a)\,\pi(n-1,\, n-r) \\ l \text{ is governed by these two rules.} \end{cases}$$

$$\text{in the second operation} \begin{cases} m_{2r} = (-1)^{\frac{2r\,(2r-1)}{2}}\,\pi(a+r-1,\, a+1)\,\pi(n-2,\, n-r) \\ m_{2r+1} = (-1)^{\frac{(2r+1)\,2r}{2}}\,\pi(a+r-1,\, a+1)\,\pi(n-2,\, n-r-1) \\ l=1 \end{cases}$$

$$\text{in the third operation} \ldots \begin{cases} m_{2r} = (-1)^{\frac{2r\,(2r-1)}{2}}\,\pi(a+r,\, a+2)\,\pi(n-3,\, n-r-1) \\ m_{2r+1} = (-1)^{\frac{(2r+1)\,2r}{2}}\,\pi(a+r,\, a+2)\,\pi(n-3,\, n-r-2) \\ l=2 \end{cases}$$

$$\text{in the fourth operation}\left\{\begin{array}{l} m_{2r}=(-1)^{\frac{2r(2r-1)}{2}}\dfrac{\pi(a+r+1,\ a+3)}{\pi(n-4,\ n-r-2)} \\[2ex] m_{2r+1}=(-1)^{\frac{(2r+1)\,2r}{2}}\dfrac{\pi(a+r+1.\ a+3)}{\pi(n-4,\ n-r-3)} \\[2ex] l=3 \end{array}\right.$$

and so on.

If is to be noted in this connection that each operation will enable us to remove certain factors and before performing the next operation, these factors are to be removed, the first column to be subtracted from the last and another factor which we can then remove from the last column is to be removed.

The Osculating Conic in Homogeneous Co-ordinates

BY

Gurudas Bhar, M.Sc.

The object of the present paper is to deduce in elegant forms the equation of the osculating conic at any point of a curve whose equation is given as a homogeneous equation of a given degree in ξ, η, ζ where ξ, η, ζ are a given system of trilinear co-ordinates satisfying a given identical relation

$$l\xi+m\eta+n\zeta=1.$$

The co-ordinates α, β, γ of any point of the given curve will be supposed expressible as functions of any arbitrary parameter t by

$$\alpha=f_1(t),\ \beta=f_2(t),\ \gamma=f_3(t)\,;$$

or more generally by

$$\alpha : \beta : \gamma = f_1(t) : f_2(t) : f_3(t).$$

In this case the absolute co-ordinates α, β, γ may be written as

$$\alpha=\frac{f_1(t)}{\theta},\ \beta=\frac{f_2(t)}{\theta},\ \gamma=\frac{f_3(t)}{\theta},$$

where

$$\theta=lf_1(t)+mf_2(t)+nf_3(t).$$

We shall say in this latter case that α, β, γ are *relative homogeneous co-ordinates* whereas in the former case we shall call them ***absolute homogeneous co-ordinates*** of a point on the curve.

1

If we take *absolute* homogeneous co-ordinates in a given system of trilinears then at a point α, β, γ of the given curve which corresponds to a certain value of t, the equation to the osculating conic can evidently

be written in the form

$$\begin{vmatrix} \xi^2 & \eta^2 & \zeta^2 & \eta\zeta & \zeta\xi & \xi\eta \\ \alpha^2 & \beta^2 & \gamma^2 & \beta\gamma & \gamma\alpha & \alpha\beta \\ D(\alpha^2) & D(\beta^2) & D(\gamma^2) & D(\beta\gamma) & D(\gamma\alpha) & D(\alpha\beta) \\ D^2(\alpha^2) & D^2(\beta^2) & D^2(\gamma^2) & D^2(\beta\gamma) & D^2(\gamma\alpha) & D^2(\alpha\beta) \\ D^3(\alpha^2) & D^3(\beta^2) & D^3(\gamma^2) & D^3(\beta\gamma) & D^3(\gamma\alpha) & D^3(\alpha\beta) \\ D^4(\alpha^2) & D^4(\beta^2) & D^4(\gamma^2) & D^4(\beta\gamma) & D^4(\gamma\alpha) & D^4(\alpha\beta) \end{vmatrix} = 0 \quad \ldots \quad (1)$$

where D^n denotes $\left(\frac{d}{dt}\right)^n$.

It is *important* to notice that we shall get the equation to the osculating conic in the *same form* even if we take α, β, γ to be *relative* co-ordinates of a point on the curve. For the second system is derivable from the first by writing $\theta\alpha$, $\theta\beta$, $\theta\gamma$ for α, β, γ respectively and all the θ's and their differential co-efficients will be eliminated from the equation on simplification.

2

Let us now take

$$U \equiv \begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix},$$

$$V \equiv \begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix}$$

$$W \equiv \begin{vmatrix} \xi & \eta & \xi \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix};$$

also let

$$\Delta_{12} \equiv \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix},$$

$$\Delta_{13} \equiv \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha''' & \beta''' & \gamma''' \end{vmatrix}$$

and generally

$$\Delta_{mn} \equiv \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^{(m)} & \beta^{(m)} & \gamma^{(m)} \\ \alpha^{(n)} & \beta^{(n)} & \gamma^{(n)} \end{vmatrix}$$

where $\alpha^{(m)} = D^m(\alpha) = \left(\dfrac{d}{dt}\right)^m(\alpha)$.

The equation (1) to the osculating conic in the new homogeneous co-ordinates U, V, W may evidently be written as

$$\begin{vmatrix} U^2 & UW & V^2 & UV & VW & W^2 \\ 0 & 0 & 0 & 0 & 0 & \Delta_{12}^3 \\ 0 & 0 & 0 & 0 & -\Delta_{12}^2 & 0 \\ 0 & \Delta_{12}^2 & 2\Delta_{12}^2 & 0 & 0 & 0 \\ 0 & \Delta_{12}\Delta_{13} & 0 & -3\Delta_{12}^2 & \Delta_{12}\Delta_{23} & 0 \\ 6\Delta_{12}^2 & \Delta_{12}\Delta_{14} & -8\Delta_{12}\Delta_{23} & -4\Delta_{12}\Delta_{13} & \Delta_{12}\Delta_{24} & 0 \end{vmatrix} = 0 \qquad (2)$$

which reduces to

$$\begin{vmatrix} U^2 & UW & V^2 & UV \\ 0 & \Delta_{12}^2 & 2\Delta_{12}^2 & 0 \\ 0 & \Delta_{12}\Delta_{13} & 0 & -3\Delta_{12}^2 \\ 6\Delta_{12}^2 & \Delta_{12}\Delta_{14} & -8\Delta_{12}\Delta_{23} & -4\Delta_{12}\Delta_{13} \end{vmatrix} = 0,$$

$$\text{or } U^2 \begin{vmatrix} 1 & 0 & 0 \\ \Delta_{13} & \Delta_{13} & 3\Delta_{12} \\ \Delta_{14} & 4\Delta_{23}+\Delta_{14} & 4\Delta_{13} \end{vmatrix} = 3\Delta_{12} \begin{vmatrix} UW & V^2 & UV \\ 1 & 2 & 0 \\ \Delta_{13} & 0 & -3\Delta_{12} \end{vmatrix}$$

or $(12\,\Delta_{12}\,\Delta_{23} - 4\Delta_{13}^2 + 3\Delta_{12}\,\Delta_{14})\,U^2$

$$= 3\Delta_{12}\,(6\Delta_{12}\,UW - 3\Delta_{12}\,V^2 + 2\Delta_{13}\,UV),$$

or $(\Delta_{13} U-3\Delta_{12} V)^2+(12\Delta_{12} \Delta_{23}-5\Delta_{13}^2+3\Delta_{12} \Delta_{14}) U^2$

$$=18\Delta_{12}^2 UW,$$

or $(\Delta_{13} U-3\Delta_{12} V)^2+\sigma U^2=18\Delta_{12}^2 UW$ ... (3)

where

$$\sigma=12\Delta_{12} \Delta_{23}-5\Delta_{13}^2+3\Delta_{12} \Delta_{14}.$$

It can also be written as

$$(\Delta_{13} U-3\Delta_{12} V)^2+\sigma \left(U-9 \frac{\Delta_{12}^2}{\sigma} W\right)^2-81\frac{\Delta_{12}^4}{\sigma} W^2=0 \quad \ldots \quad (4)$$

From (4) it is evident that

$$\left.\begin{array}{l} \Delta_{13} U-3\Delta_{12} V=0, \\ U-9 \dfrac{\Delta_{12}^2}{\sigma} W=0, \\ W=0 \end{array}\right\} \quad (5)$$

form a self-conjugate triangle of the osculating conic.

Now the tangent to the curve and therefore also to the osculating conic at α, β, γ is evidently

$$U \equiv \begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix} =0 \quad \ldots \quad (6)$$

whether α, β, γ be absolute or relative homogeneous co-ordinates.

Again if we take α, β, γ to be *absolute* homogeneous co-ordinates, then

$$l\alpha+m\beta+n\gamma=1;$$

therefore

$$\left.\begin{array}{l} l\alpha'+m\beta'+n\gamma'=0, \\ l\alpha''+m\beta''+n\gamma''=0 \end{array}\right\} \quad \ldots \quad (7)$$

The equation to the line at infinity is

$$l\xi+m\eta+n\zeta=0 \quad \ldots \quad (8)$$

Eliminating l, m, n from (7) and (8) we have for the equation of the line at infinity

$$W \equiv \begin{vmatrix} \xi & \eta & \zeta \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} = 0 \qquad \dots (9)$$

The pole of the line at infinity is the *centre* of the osculating conic which is determined by the intersection of

$$\Delta_{13}\, U - 3\Delta_{12}\, V = 0$$

and

$$U - 9\,\frac{\Delta_{12}^2}{\sigma}\, W = 0.$$

But $U=0$ is the tangent to the osculating conic at the point of contact α, β, γ. Therefore

$$U - 9\,\frac{\Delta_{12}^2}{\sigma}\, W = 0 \qquad \dots (10)$$

is the diameter of the osculating conic parallel to the tangent.

Again $U=0$ and $V=0$ both pass through α, β, γ. Therefore

$$\Delta_{13}\, U - 3\Delta_{12}\, V = 0 \qquad \dots (11)$$

is the diameter of the osculating conic passing through the point of contact. It is the *axis of aberrancy* of the given curve at α, β, γ.

3

The co-ordinates of the centre are given by the intersection of

$$\Delta_{13}\, U - 3\Delta_{12}\, V = 0,$$

and

$$U - 9\,\frac{\Delta_{12}^2}{\sigma}\, W = 0;$$

that is by

$$\frac{U}{9\Delta_{12}^2} = \frac{V}{3\Delta_{12}\,\Delta_{13}} = \frac{W}{\sigma},$$

or by

$$\frac{\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}}{9\Delta_{12}^2} = \frac{\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix}}{3\Delta_{12}\,\Delta_{13}} = \frac{\begin{vmatrix} \xi & \eta & \zeta \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix}}{\sigma} \quad \ldots \quad (12)$$

Now from (7) we have

$$\frac{l}{\beta'\gamma''-\beta''\gamma'} = \frac{m}{\gamma'\alpha''-\gamma''\alpha'} = \frac{n}{\alpha'\beta''-\alpha''\beta'}$$

$$= \frac{l\alpha+m\beta+n\gamma}{\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix}} = \frac{1}{\Delta_{12}}.$$

Hence

$$\left.\begin{aligned} \beta'\gamma''-\beta''\gamma' &= l\Delta_{12}, \\ \gamma'\alpha''-\gamma''\alpha' &= m\Delta_{12}, \\ \alpha'\beta''-\alpha''\beta' &= n\Delta_{12}. \end{aligned}\right\} \quad \ldots \quad (13)$$

Therefore the determinant

$$\begin{vmatrix} \xi & \eta & \zeta \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} = (\beta'\gamma''-\beta''\gamma')\,\xi + (\gamma'\alpha''-\gamma''\alpha')\,\eta + (\alpha'\beta''-\alpha''\beta')\,\zeta,$$

$$= \Delta_{12}\,(l\xi+m\eta+n\zeta), \text{ by (13)},$$

$$= \Delta_{12} \quad \ldots \quad (14)$$

Each of the ratios in (12) is therefore equal to $\frac{\Delta_{12}}{\sigma}$ and the co-ordinates of the centre are determined by

$$(\beta\gamma'-\beta'\gamma)\,\xi + (\gamma\alpha'-\gamma'\alpha)\,\eta + (\alpha\beta'-\alpha'\beta)\,\zeta = 9\,\frac{\Delta_{12}^3}{\sigma}.$$

$$(\beta\gamma''-\beta''\gamma)\,\xi + (\gamma\alpha''-\gamma''\alpha)\,\eta + (\alpha\beta''-\alpha''\beta)\,\zeta = 3\,\frac{\Delta_{12}^2\,\Delta_{13}}{\sigma}.$$

$$(\beta'\gamma''-\beta''\gamma')\,\xi + (\gamma'\alpha''-\gamma''\alpha')\eta + (\alpha'\beta''-\alpha''\beta')\,\zeta = \Delta_{12},$$

whence we get for the co-ordinates of the centre

$$\left.\begin{aligned} \xi&=\alpha-\frac{3\Delta_{12}\,(\Delta_{13}\,\alpha'-3\Delta_{12}\,\alpha'')}{\sigma}, \\ \eta&=\beta-\frac{3\Delta_{12}\,(\Delta_{13}\,\beta'-3\Delta_{12}\,\beta'')}{\sigma}, \\ \zeta&=\gamma-\frac{3\Delta_{12}\,(\Delta_{13}\,\gamma'-3\Delta_{12}\,\gamma'')}{\sigma}. \end{aligned}\right\} \qquad \dots \quad (15)$$

4

If the osculating conic passes through six consecutive points at α, β, γ, we must have

$$\begin{vmatrix} \alpha^2 & \beta^2 & \gamma^2 & \beta\gamma & \gamma\alpha & \alpha\beta \\ D(\alpha^2) & D(\beta^2) & D(\gamma^2) & D(\beta\gamma) & D(\gamma\alpha) & D(\alpha\beta) \\ D^2(\alpha^2) & D^2(\beta^2) & D^2(\gamma^2) & D^2(\beta\gamma) & D^2(\gamma\alpha) & D^2(\alpha\beta) \\ D^3(\alpha^2) & D^3(\beta^2) & D^3(\gamma^2) & D^3(\beta\gamma) & D^3(\gamma\alpha) & D^3(\alpha\beta) \\ D^4(\alpha^2) & D^4(\beta^2) & D^4(\gamma^2) & D^4(\beta\gamma) & D^4(\gamma\alpha) & D^4(\alpha\beta) \\ D^5(\alpha^2) & D^5(\beta^2) & D^5(\gamma^2) & D^5(\beta\gamma) & D^5(\gamma\alpha) & D^5(\alpha\beta) \end{vmatrix}=0. \qquad \dots \quad (16)$$

From this we derive, just as (2) is derived from (1), the relation

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & \Delta_{12}^2 \\ 0 & 0 & 0 & 0 & -\Delta_{12}^2 & 0 \\ 0 & \Delta_{12}^2 & 2\Delta_{12}^2 & 0 & 0 & 0 \\ 0 & \Delta_{12}\Delta_{13} & 0 & -3\Delta_{12}^2 & \Delta_{12}\Delta_{23} & 0 \\ 6\Delta_{12}^2 & \Delta_{12}\Delta_{14} & -8\Delta_{12}\Delta_{23} & -4\Delta_{12}\Delta_{13} & \Delta_{12}\Delta_{24} & 0 \\ 20\Delta_{12}\Delta_{13} & \Delta_{12}\Delta_{15}+10\Delta_{12}\Delta_{123} & -10\Delta_{12}\Delta_{24} & -5\Delta_{12}\Delta_{14}+10\Delta_{12}\Delta_{23} & \Delta_{12}\Delta_{25} & 0 \end{vmatrix}=0$$

where

$$\Delta_{123}=\begin{vmatrix} \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \\ \alpha''' & \beta''' & \gamma''' \end{vmatrix};$$

it reduces to

$$20\Delta_{12}\Delta_{13}\begin{vmatrix} 1 & 0 & 0 \\ \Delta_{13} & \Delta_{13} & 3\Delta_{12} \\ \Delta_{14} & \Delta_{14}+4\Delta_{23} & 4\Delta_{13} \end{vmatrix}.$$

$$=3\Delta_{12}\begin{vmatrix} \Delta_{12}\Delta_{15}+10\Delta_{12}\Delta_{123} & -10\Delta_{12}\Delta_{24} & -5\Delta_{12}\Delta_{14}+10\Delta_{12}\Delta_{23} \\ 1 & 2 & 0 \\ \Delta_{13} & 0 & -3\Delta_{12} \end{vmatrix}$$

On further simplification we obtain

$$40\Delta_{13}^3-45\Delta_{12}\Delta_{13}\Delta_{14}+9\Delta_{12}^2\Delta_{15}-90\Delta_{12}\Delta_{13}\Delta_{23}$$
$$+45\Delta_{12}^2\Delta_{24}+90\Delta_{12}^2\Delta_{123}=0 \quad \dots \quad (17)$$

which is therefore the condition that α, β, γ may be a sextactic point on the curve.

If α, β, γ be *absolute* homogeneous co-ordinates $\Delta_{123}=0$ and the above condition reduces to

$$40\Delta_{13}^3-45\Delta_{12}\Delta_{13}\Delta_{14}+9\Delta_{12}^2\Delta_{15}-90\Delta_{12}\Delta_{13}\Delta_{23}$$
$$+45\Delta_{12}^2\Delta_{24}=0 \quad \dots \quad (18)$$

5

Suppose now α, β, γ are the relative homogeneous co-ordinates of a point on the curve. Let us write

$$\Theta=L\xi+M\eta+N\zeta$$

where L, M, N are any constants ; also let

$$\theta=L\alpha+M\beta+N\gamma,$$

so that

$$\theta'=L\alpha'+M\beta'+N\gamma',$$

$$\theta''=L\alpha''+M\beta''+N\gamma''$$

and generally

$$\theta^{(n)}=L\alpha^{(n)}+M\beta^{(n)}+N\gamma^{(n)}.$$

The equation (1) to the osculating conic can now be written as

$$\begin{vmatrix} U^2 & U\Theta & V^2 & UV & V\Theta & \Theta^2 \\ 0 & 0 & 0 & 0 & 0 & \theta^2 \\ 0 & 0 & 0 & 0 & -\theta\Delta_{12} & 2\theta\theta' \\ 0 & \theta\Delta_{12} & 2\Delta_{12}^2 & 0 & -2\theta\Delta_{12} & 2\theta'^2+2\theta\theta' \\ 0 & 3\theta'\Delta_{12}+\theta\Delta_{13} & 0 & -3\Delta_{12}^2 & \theta\Delta_{23}-3\theta''\Delta_{12} & 6\theta'\theta''+2\theta\theta' \\ 6\Delta_{12}^2 & 6\theta''\Delta_{12}+4\theta'\Delta_{13}+\theta\Delta_{14} & -8\Delta_{12}\Delta_{23} & -4\Delta_{12}\Delta_{13} & \theta\Delta_{24}+4\theta'\Delta_{23}-4\theta'''\Delta_{12} & 6\theta''^2+8\theta'\theta'''+2\theta\theta^{iv} \end{vmatrix}=0 \quad (19)$$

On simplification this reduces to

$$(12\theta\Delta_{12}\Delta_{23}+3\theta\Delta_{12}\Delta_{14}-4\theta\Delta_{13}^2+18\theta''\Delta_{12}^2)U^2$$

$$+9\theta\Delta_{12}^2 V^2-18\Delta_{12}^3 U\Theta-(18\theta'\Delta_{12}^2+6\theta\Delta_{12}\Delta_{13})UV=0,$$

or $\{(\theta\Delta_{13}+3\theta'\Delta_{12})U-3\theta\Delta_{12}V\}^2$

$$+\{(12\Delta_{12}\Delta_{23}-5\Delta_{13}^2+3\Delta_{12}\Delta_{14})\theta^2-6\theta\theta'\Delta_{12}\Delta_{13}$$

$$+9\Delta_{12}^2(2\theta\theta''-\theta'^2)\}U^2$$

$$=18\theta\Delta_{12}^3 U\Theta,$$

or $\{(\theta\Delta_{13}+3\theta'\Delta_{12})U-3\theta\Delta_{12}V\}^2+U^2\Sigma=18\theta\Delta_{12}^3 U\Theta,$ ... (20)

where

$$\Sigma=(12\Delta_{12}\Delta_{23}-5\Delta_{13}^2+3\Delta_{12}\Delta_{14})\theta^2-6\theta\theta'\Delta_{12}\Delta_{13}$$

$$+9\Delta_{12}^2(2\theta\theta''-\theta'^2)$$

$$=\sigma\theta^2-6\theta\theta'\Delta_{12}\Delta_{13}+9\Delta_{12}^2(2\theta\theta'-\theta'^2) \qquad \text{... (21)}$$

If $\Sigma=0$, the equation to the osculating conic reduces to

$$\{(\theta\Delta_{13}+3\theta'\Delta_{12})U-3\theta\Delta_{12}V\}^2=18\theta\Delta_{12}^3 U\Theta \qquad \text{... (22)}$$

The form of the equation shows that $U=0$ as well as $\Theta-0$ are tangential to the osculating conic and

$$(\theta\Delta_{13}+3\theta'\Delta_{13})U=3\theta\Delta_{12}V$$

is the chord of contact. Hence the tangential equation of the osculating conic to the given curve at α, β, γ is

$$\Sigma=0 \qquad \text{... (23)}$$

It may be observed that $\Sigma=0$ is a homogeneous equation of the second degree in L, M, N for θ, θ' and θ'' are linear expressions of the first degree in L, M, N.

6

We now proceed to obtain the same results by a direct transformation of the results given by Prof. S. Mukhopadhyaya in his "General Theory of Osculating Conics"—Second paper. [*Journal and Proceedings, Asiatic Society of Bengal (New series), Vol. IV, No. 10, 1908*].

Suppose ξ, η, ζ are absolute homogeneous co-ordinates in the system of trilinears

$$\left.\begin{aligned} \xi&=AX+BY+C=\lambda\,(p-X\cos\theta-Y\sin\theta)=0,\\ \eta&=A'X+B'Y+C'=\mu\,(q-X\cos\phi-Y\sin\phi)=0,\\ \zeta&=A''X+B''Y+C''=\nu\,(r-X\cos\psi-Y\sin\psi)=0. \end{aligned}\right\} \quad \ldots \quad (24)$$

The identical relation satisfied by ξ, η, ζ is

$$l\xi+m\eta+n\zeta=1 \quad \ldots \quad (25)$$

where

$$l=\frac{a}{2\lambda\Delta},\ m=\frac{b}{2\mu\Delta},\ n=\frac{c}{2\nu\Delta} \quad \ldots \quad (26)$$

a, b, c being the sides and Δ the area of the fundamental triangle.

We shall denote by ξ, η, ζ the current homogeneous co-ordinates in this system and by α, β, γ the co-ordinates of a given point. Similarly X, Y shall denote the current co-ordinates in a rectangular cartesian system and x, y the co-ordinates of a given point in this system.

Let

$$D=\begin{vmatrix} \xi & \eta & \zeta \\ A & A' & A'' \\ B & B' & B'' \end{vmatrix}$$

then D has a constant value; for

$$D=\xi\,(A'B''-A''B')+\eta\,(A''B-AB'')+\zeta\,(AB'-A'B)$$

$$=\mu\nu\xi\,(\cos\phi\sin\psi-\cos\psi\sin\phi)+\nu\lambda\eta\,(\cos\psi\sin\theta-\cos\theta\sin\psi)$$

$$+\lambda\mu\zeta\,(\cos\theta\sin\phi-\cos\phi\sin\theta)$$

$$=\mu\nu\xi\sin\widehat{bc}+\nu\lambda\eta\sin\widehat{ca}+\lambda\mu\zeta\sin\widehat{ab},$$

$\widehat{bc}, \widehat{ca}, \widehat{ab}$ being the angles between b and c, c and a, a and b respectively. If now ρ be the radius of the circum-circle of the fundamental triangle, we have

$$D=\frac{\lambda\mu\nu}{2\rho}\left\{\frac{a\xi}{\lambda}+\frac{b\eta}{\mu}+\frac{c\zeta}{\nu}\right\}$$

$$=\lambda\mu\nu\frac{\Delta}{\rho}, \text{ by (25) and (26).}$$

It is therefore evident that

$$\begin{vmatrix} \alpha & \beta & \gamma \\ A & A' & A'' \\ B & B' & B'' \end{vmatrix} = \begin{vmatrix} \xi & \eta & \zeta \\ A & A' & A'' \\ B & B' & B'' \end{vmatrix} = D. \qquad \ldots \quad (27)$$

From (24) it is clear that C, C′, C″ are the trilinear co-ordinates of the point which is the origin of co-ordinates in the cartesian system. Hence also

$$\begin{vmatrix} A & A' & A'' \\ B & B' & B'' \\ C & C' & C'' \end{vmatrix} = D. \qquad \ldots \quad (28)$$

7

In what follows we shall make frequent use of the following determinant identity :[1]

$$\begin{vmatrix} f & g & h \\ b & b' & b'' \\ c & c' & c'' \end{vmatrix} \begin{vmatrix} a & a' & a'' \\ l & m & n \\ p & q & r \end{vmatrix} + \begin{vmatrix} a & a' & a'' \\ f & g & h \\ c & c' & c'' \end{vmatrix} \begin{vmatrix} b & b' & b'' \\ l & m & n \\ p & q & r \end{vmatrix} + \begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ f & g & h \end{vmatrix} \begin{vmatrix} c & c' & c'' \\ l & m & n \\ p & q & r \end{vmatrix}$$

[1] The identities (29) and (30) are but particular cases of the following mor general Determinantal Identity :

$$\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{r1} & a_{r2} & \ldots & a_{rn} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} \begin{vmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ b_{r1} & b_{r2} & \ldots & b_{rn} \\ \ldots & \ldots & \ldots & \ldots \\ b_{n1} & b_{n2} & \ldots & b_{nn} \end{vmatrix}$$

$$= \sum_{m=1}^{m-n} \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ b_{11} & b_{12} & \ldots & b_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} \begin{vmatrix} a_{m1} & a_{m2} & \ldots & a_{mn} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ b_{m1} & b_{m2} & \ldots & b_{mn} \\ \ldots & \ldots & \ldots & \ldots \\ b_{n1} & b_{n2} & \ldots & b_{nn} \end{vmatrix}$$

where the summation consists of n products which are derived from the product of the two determinants on the left-hand side by interchanging the first row of the second detetminant with the successive rows of the first determinant.

$$=\begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{vmatrix}\begin{vmatrix} f & g & h \\ l & m & n \\ p & q & r \end{vmatrix} \qquad \ldots \quad (29)$$

which is easily proved.

If in (29) we replace c, c', c'' by l, m, n respectively we get

$$\begin{vmatrix} f & g & h \\ b & b' & b'' \\ l & m & n \end{vmatrix}\begin{vmatrix} a & a' & a'' \\ l & m & n \\ p & q & r \end{vmatrix}+\begin{vmatrix} a & a' & a'' \\ f & g & h \\ l & m & n \end{vmatrix}\begin{vmatrix} b & b' & b'' \\ l & m & n \\ p & q & r \end{vmatrix}=\begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ l & m & n \end{vmatrix}\begin{vmatrix} f & g & h \\ l & m & n \\ p & q & r \end{vmatrix}$$

which rewritten is

$$\begin{vmatrix} l & m & n \\ a & a' & a'' \\ f & g & h \end{vmatrix}\begin{vmatrix} l & m & n \\ b & b' & b'' \\ p & q & r \end{vmatrix}-\begin{vmatrix} l & m & n \\ a & a' & a'' \\ p & q & r \end{vmatrix}\begin{vmatrix} l & m & n \\ b & b' & b'' \\ f & g & h \end{vmatrix}$$

$$=\begin{vmatrix} l & m & n \\ a & a' & a'' \\ b & b' & b'' \end{vmatrix}\begin{vmatrix} l & m & n \\ f & g & h \\ p & q & r \end{vmatrix} \qquad \ldots \quad (30)$$

The identity in this latter form will be most useful.

8

From equations (24) we obtain

$$X=\frac{1}{D}\begin{vmatrix} \zeta & \eta & \zeta \\ B & B' & B'' \\ C & C' & C'' \end{vmatrix} \qquad \ldots \quad (31)$$

$$Y=\frac{1}{D}\begin{vmatrix} \xi & \eta & \zeta \\ C & C' & C'' \\ A & A' & A'' \end{vmatrix} \qquad \ldots \quad (32)$$

From (31) by differentiation we get

$$X'=\frac{1}{D^2}\left\{\begin{vmatrix} \xi' & \eta' & \zeta' \\ B & B' & B'' \\ C & C' & C'' \end{vmatrix}\begin{vmatrix} \xi & \eta & \zeta \\ A & A' & A'' \\ B & B' & B'' \end{vmatrix}-\begin{vmatrix} \xi & \eta & \zeta \\ B & B' & B'' \\ C & C' & C'' \end{vmatrix}\begin{vmatrix} \xi' & \eta' & \zeta' \\ A & A' & A'' \\ B & B' & B'' \end{vmatrix}\right\},$$

$$=\frac{1}{D^3}\begin{vmatrix} A & A' & A'' \\ B & B' & B'' \\ C & C' & C'' \end{vmatrix}\begin{vmatrix} B & B' & B'' \\ \xi & \eta & \zeta \\ \xi' & \eta' & \zeta \end{vmatrix} \text{ by (30),}$$

$$=\frac{1}{D}\begin{vmatrix} B & B' & B'' \\ \xi & \eta & \zeta \\ \xi' & \eta' & \zeta' \end{vmatrix} \text{ by (28)} \qquad \dots \quad (33)$$

Similarly,

$$Y'=-\frac{1}{D}\begin{vmatrix} A & A' & A'' \\ \xi & \eta & \zeta \\ \xi' & \eta' & \zeta' \end{vmatrix}. \qquad \dots \quad (34)$$

Again differentiating X' and Y' we have

$$X''=\frac{1}{D}\begin{vmatrix} B & B' & B'' \\ \xi & \eta & \zeta \\ \xi'' & \eta'' & \zeta'' \end{vmatrix}, \qquad \dots \quad (35)$$

$$Y''=-\frac{1}{D}\begin{vmatrix} A & A' & A'' \\ \xi & \eta & \zeta \\ \xi'' & \eta'' & \zeta'' \end{vmatrix}. \qquad \dots \quad (36)$$

From (35) by further differentiation

$$X'''=\frac{1}{D}\left\{\begin{vmatrix} B & B' & B'' \\ \xi' & \eta' & \zeta' \\ \xi'' & \eta'' & \zeta'' \end{vmatrix}+\begin{vmatrix} B & B' & B'' \\ \xi & \eta & \zeta \\ \xi''' & \eta''' & \zeta''' \end{vmatrix}\right\}$$

Now since

$$l\xi+m\eta+n\zeta=1,$$

we have

$$l\xi'+m\eta'+n\zeta'=0,$$

$$l\rho''+m\eta''+n\zeta''=0,$$

$$\therefore \frac{l}{\eta'\zeta''-\eta''\zeta'}=\frac{m}{\zeta'\xi''-\zeta''\xi'}=\frac{n}{\xi'\eta''-\xi''\eta'}=\frac{1}{k} \text{ say.} \qquad \dots \quad (37)$$

Again

$$A'B''-A''B'=\mu\nu\{\cos\phi\sin\psi-\cos\psi\sin\phi\}=\mu\nu\sin\widehat{bc}$$

$$\left.\begin{aligned} &=\frac{a\mu\nu}{2\rho}=\lambda\mu\nu\frac{\Delta}{\rho}l, \\ A''B-AB''&=\frac{b\nu\lambda}{2\rho}=\lambda\mu\nu\frac{\Delta}{\rho}m, \\ AB'-A'B&=\frac{c\lambda\mu}{2\rho}=\lambda\mu\nu\frac{\Delta}{\rho}, \end{aligned}\right\} \quad \ldots \quad (38)$$

The determinant

$$\begin{vmatrix} B & B' & B'' \\ \xi' & \eta' & \zeta' \\ \xi'' & \eta'' & \zeta'' \end{vmatrix} = B(\eta'\zeta''-\eta''\zeta')+B'(\zeta'\xi''-\zeta''\xi')+B''(\xi'\eta''-\xi''\eta')$$

$$=\frac{k}{\lambda\mu\nu}\cdot\frac{\rho}{\Delta}\{B(A'B''-A''B')+B'(A''B-AB'')$$

$$+B''(AB'-A'B)\} \text{ from (37) and (38),}$$

$$=0$$

Hence

$$X'''=\frac{1}{D}\begin{vmatrix} B & B' & B'' \\ \xi & \eta & \zeta \\ \xi''' & \eta''' & \zeta''' \end{vmatrix} \quad \ldots \quad (39)$$

Similarly

$$Y'''=-\frac{1}{D}\begin{vmatrix} A & A' & A'' \\ \xi & \eta & \zeta \\ \xi''' & \eta''' & \zeta''' \end{vmatrix} \quad \ldots \quad (40)$$

Proceeding as above we can show that

$$X(n)=\frac{1}{D}\begin{vmatrix} B & B' & B'' \\ \xi & \eta & \zeta \\ \xi(n) & \eta(n) & \zeta(n) \end{vmatrix} \quad \ldots \quad (41)$$

$$Y^{(n)}=-\frac{1}{D}\begin{vmatrix} A & A' & A'' \\ \xi & \eta & \zeta \\ \xi^{(n)} & \eta^{(n)} & \zeta^{(n)} \end{vmatrix}, \qquad \ldots \quad (42)$$

We shall now make use of those transformation formulae to find the values of the functions P, Q, R, S, T, Q_1, R′, S′ which are defined below:

We have

$$x^{(m)}y^{(n)}-x^{(n)}y^{(m)}$$

$$=\frac{1}{D^2}\left\{\begin{vmatrix} \alpha & \beta & \gamma \\ A & A' & A'' \\ \alpha^{(m)} & \beta^{(m)} & \gamma^{(m)} \end{vmatrix}\begin{vmatrix} \alpha & \beta & \gamma \\ B & B' & B'' \\ \alpha^{(n)} & \beta^{(n)} & \gamma^{(n)} \end{vmatrix}\right.$$

$$\left.-\begin{vmatrix} \alpha & \beta & \gamma \\ A & A' & A'' \\ \alpha^{(n)} & \beta^{(n)} & \gamma^{(n)} \end{vmatrix}\begin{vmatrix} \alpha & \beta & \gamma \\ B & B' & B'' \\ \alpha^{(m)} & \beta^{(m)} & \gamma^{(m)} \end{vmatrix}\right\},$$

$$=\frac{1}{D^2}\begin{vmatrix} \alpha & \beta & \gamma \\ A & A' & A'' \\ B & B' & B'' \end{vmatrix}\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^{(m)} & \beta^{(m)} & \gamma^{(m)} \\ \alpha^{(n)} & \beta^{(n)} & \gamma^{(n)} \end{vmatrix} \text{ by (30)}$$

$$=\frac{\Delta_{mn}}{D}.$$

Hence

$$\left.\begin{aligned}
Q &= x'y'' - x''y' = \frac{\Delta_{12}}{D},\\
R &= x'y''' - x'''y' = \frac{\Delta_{13}}{D},\\
S &= x'y^{iv} - x^{iv}y' = \frac{\Delta_{14}}{D},\\
T &= x'y^{v} - x^{v}y' = \frac{\Delta_{15}}{D},\\
R &= x''y''' - x'''y'' = \frac{\Delta_{23}}{D},\\
S' &= x''y^{iv} - x^{iv}y'' = \frac{\Delta_{24}}{D},
\end{aligned}\right\} \qquad \ldots \quad (43)$$

$$P = x'^2 + y'^2 = \frac{1}{D^2}\left\{ \begin{vmatrix} A & A' & A'' \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}^2 + \begin{vmatrix} B & B' & B'' \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}^2 \right\} \ldots \quad (44)$$

$$Q_1 = x'x'' + y'y'' = \frac{1}{D^2}\left\{ \begin{vmatrix} A & A' & A'' \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix} \begin{vmatrix} A & A' & A'' \\ \alpha & \beta & \gamma \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} \right.$$

$$\left. + \begin{vmatrix} B & B' & B'' \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix} \begin{vmatrix} B & B' & B'' \\ \alpha & \beta & \gamma \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} \right\} \ldots \quad (45)$$

Now the expression

$$(Y-y)x' - (X-x)y'$$

or

$$Xx' - Xy' - x'y + xy'$$

becomes on transformation

$$=\frac{1}{D^2}\left\{\begin{vmatrix}\xi & \eta & \zeta\\ C & C' & C''\\ A & A' & A''\end{vmatrix}\begin{vmatrix}B & B' & B''\\ \alpha & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}+\begin{vmatrix}\xi & \eta & \zeta\\ B & B' & B''\\ C & C' & C''\end{vmatrix}\begin{vmatrix}A & A' & A''\\ \alpha & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}\right.$$

$$\left.+\begin{vmatrix}A & A' & A''\\ \alpha & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}\begin{vmatrix}B & B' & B''\\ \alpha & \beta & \gamma\\ C' & C' & C'\end{vmatrix}-\begin{vmatrix}A & A' & A''\\ \alpha & \beta & \gamma\\ C & C' & C''\end{vmatrix}\begin{vmatrix}B & B' & B''\\ \alpha & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}\right\},$$

$$=\frac{1}{D^2}\left\{\begin{vmatrix}\xi & \eta & \zeta\\ B & B' & B''\\ C & C' & C''\end{vmatrix}\begin{vmatrix}A & A' & A''\\ \alpha & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}+\begin{vmatrix}\xi & \eta & \zeta\\ C & C' & C''\\ A & A' & A''\end{vmatrix}\begin{vmatrix}B & B' & B''\\ \alpha & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}\right.$$

$$\left.+\begin{vmatrix}\alpha & \beta & \gamma\\ A & A' & A''\\ B & B' & B''\end{vmatrix}\begin{vmatrix}C & C' & C''\\ \alpha & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}\right\}\text{ by (30)},$$

$$=\frac{1}{D^2}\left\{\begin{vmatrix}\xi & \eta & \zeta\\ B & B' & B''\\ C & C' & C''\end{vmatrix}\begin{vmatrix}A & A' & A''\\ & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}+\begin{vmatrix}A & A' & A''\\ \xi & \eta & \zeta\\ C & C' & C''\end{vmatrix}\begin{vmatrix}B & B' & B''\\ \alpha & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}\right.$$

$$\left.+\begin{vmatrix}A & A' & A''\\ B & B' & B''\\ \xi & \eta & \zeta\end{vmatrix}\begin{vmatrix}C & C' & C''\\ \alpha & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}\right.\text{ by (27)},$$

$$=\frac{1}{D^2}\left\{\begin{vmatrix}A & A' & A''\\ B & B' & B''\\ C & C' & C''\end{vmatrix}\right.\begin{vmatrix}\xi & \eta & \zeta\\ \alpha & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}\text{ by (29)},$$

$$=\frac{1}{D}\begin{vmatrix}\xi & \eta & \zeta\\ \alpha & \beta & \gamma\\ \alpha' & \beta' & \gamma'\end{vmatrix}\text{ by (28).} \qquad \ldots \quad (46)$$

Similarly the expression

$$(Y-y)x''-(X-x)y''$$

or $$Yx''-Xy''-x''y+xy''$$

reduces on transformation to

$$\frac{1}{D}\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix}. \qquad \dots \quad (47)$$

9

The equation to the osculating parabola at a point x, y on the curve is

$$\{(Y-y)(3Qx''-Rx')-(X-x)(3Qy''-Ry')\}^2$$
$$=18Q^3\{(Y-y)x'-(X-x)y'\},$$

[See S. Mukhopadhyaya—A General Theory of Osculating Conics, Second paper : *Journal and Proceedings, Asiatic Society of Bengal (New Series) Vol. IV, No. 10, 1908.*]

or, $$\{3Q[(Y-y)x''-(X-x)y'']-R[(Y-y)x'-(X-x)y']\}^2$$
$$=18Q^3\{(Y-y)x'-(X-x)y'\} ;$$

this on being transformed is

$$3\Delta_{12}\left\{\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} - \Delta_{13}\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}\right\}^2 = 18\Delta_{12}^3\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}.$$

$$\text{or, } 3\Delta_{12}\left\{\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} - \Delta_{13}\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}\right\}^2$$

$$=18\Delta_{12}^2\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}\begin{vmatrix} \xi & \eta & \zeta \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} \text{ by (14)} \qquad \dots \quad (48)$$

or $$(\Delta_{13}U-3\Delta_{12}V)^2=18\Delta_{12}^2UW.$$

Again the equation to the osculating conic at a point (x,y) on the curve is

$$\{(Y-y)(3Qx''-Rx')-(X-x)(3Qy''-Ry')\}^2$$
$$+(3QS-5R^2+12QR')\{(Y-y)x'-(X-x)y'\}^2$$
$$=18Q^3\{(Y-y)x'-(X-x)y'\},$$

[See, A General Theory of Osculating Conics, *loc. cit.* (52).]

or,
$$\{3Q[(Y-y)x''-(X-x)y'']-R[(Y-y)x'-(X-x)y']\}^2$$
$$+(3QS-5R^2+12QR')\{Y-y)x'-(X-x)y'\}^2$$
$$=18Q^3\{(Y-y)x'-(X-x)y'\}.$$

This transforms into

$$3\Delta_{12}\left\{\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} - \Delta_{13}\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}\right\}^2$$

$$+(12\Delta_{12}\Delta_{23}-5\Delta_{13}^2+3\Delta_{12}\Delta_{14})\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}^2$$

$$=18\Delta_{12}^3\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix},$$

or
$$3\Delta_{12}\left\{\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} - \Delta_{13}\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}\right\}^2$$

$$+(12\Delta_{12}\Delta_{23}-5\Delta_{13}^2+3\Delta_{12}\Delta_{14})\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}^2$$

$$=18\Delta_{12}^2\begin{vmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}\begin{vmatrix} \xi & \eta & \zeta \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} \text{ by (14), (49)}$$

or $$(\Delta_{13}U-3\Delta_{12}V)^2+\sigma U^2=18\Delta_{12}^2 UW,$$

which is the same as equation (3).

Finally if the osculating conic has a six-pointic contact at α, β, γ we must have

$$40R^3-45QRS+9Q^2T-90QRR'+45Q^2S'=0,$$

[See, A General Theory of Osculating Conics, *loc. cit.*] which transforms into relation (18) of § 4.

This paper was written at the suggestion and under the guidance of Prof. S. Mukhopadhyaya to whom my best thanks are due.

On some Laws of Central Force.

Part I.

BY

N. M. Basu, M.Sc.

1. The law of Central force under which a particle can describe a a conic, whatever may be the conditions of projection, has formed the subject of investigation by mathematicians from very early times. Among the contributors to the subject may be mentioned the names of Newton,[1] Sir W. R. Hamilton,[2] Villarceau,[3] Darboux,[4] Halphen, Glaisher,[5] Hirayama,[6] Appell[7] and others. These mathematicians confined themselves to the consideration of the law of force as being a function of the position of the particle relative to the centre of force.

2. The problem of determining the law of force when it depends on the velocity as well as on the position of the moving particle appears to have been first solved by M. Paul J. Suchar[8] who has shown that, besides the two well-known laws of force depending on the position, there are six and only six other laws depending on the velocity as well. These six laws are :—

(*i*) $\mu(a\dot{x}_1+b\dot{y}_1+c)^3 r_1$, (*ii*) $\mu(a\dot{x}_1^2+2b\dot{x}_1\dot{y}_1+c\dot{y}_1^2)^2 r_1$,

(*iii*) $\mu\dot{y}_1^3(ax_1+by_1+c)^{-3} r_1$ (*iv*) $\mu\dot{y}_1^3(ay_1^2+2by_1+c)^{-\frac{3}{2}} r_1$,

v) $\mu y_1^{-3}(a\dot{x}_1+b\dot{y}_1+c)^3 r_1$, (*vi*) $\mu y_1^{-3}(a\dot{y}_1^2+2b\dot{y}_1+c)^{\frac{3}{2}} r_1$,

and the laws of force previously known are

(*vii*) $\mu(ax_1+by_1+c)^{-3} r_1$, (*viii*) $\mu(ax_1^2+2bx_1y_1+cy_1^2)^{-\frac{3}{2}} r_1$

[1] First book of the Principia.

[2] Proceedings of the Irish Academy, Vol. 3, 1846.

[3] Connaissance des Temps, 1852.

[4] Comptes Rendus, Vol. 84, 1877.

[5] Monthly Notices of the Royal Astronomical Society, Vol. 39, 1878.

[6] Gould's Astronomical Journal, 1889.

[7] American Journal of Mathematics, Vol. 13, 1891.

[8] Nouvelles Annales de Mathematiques, Vol. 6, Series, 4, 1906.

where x_1, y_1, are the co-ordinates and r_1 the distance from the origin of the moving particle at time t_1, μ, a, b, c are arbitrary constants and $\dot{x}_1, \dot{y}_1$ stand for $\frac{dx_1}{dt_1}$ and $\frac{dy_1}{dt_1}$ respectively.

3. In obtaining these laws Suchar has made use of a very important theorem established by himself.[1] Using the transformation,

$$\dot{x}=\frac{dx}{dt}=x', \quad \dot{y}=\frac{dy}{dt}=y'. \quad \frac{dt'}{dt}=\mathrm{F}(x, y, \dot{x}, \dot{y})$$

where (x,y) are the co-ordinates, at time t, of a particle moving under a central force at the origin whose acceleration per unit mass is $r\mathrm{F}\ (x,y,\dot{x},\dot{y})$, where $r=(x^2+y^2)^{\frac{1}{2}}$, we obtain

$$\dot{x}' = \frac{dx'}{dt'} = \dot{x}, \quad \dot{y}' = \frac{dy'}{dt'} = \dot{y},$$

whence we further obtain

$$\ddot{x}' = \frac{d^2x'}{dt'^2} - \frac{x'}{\mathrm{F}(x, y, \dot{x}, \dot{y})} = \frac{x'}{\mathrm{F}(\dot{x}',\dot{y}', x',y')}$$

and

$$\ddot{y}' = \frac{d^2y'}{dt'^2} = \frac{y'}{\mathrm{F}\ (x,y,\dot{x}',\dot{y}')} = \frac{y'}{\mathrm{F}(\dot{x}',\dot{y}',x',y')}.$$

It thus follows that if (x',y') denotes the position of a second moving particle at the time t', this particle moves under a central force at the origin whose acceleration per unit mass is $r'/\mathrm{F}(\dot{x}',\dot{y}',x',y'), r'$ being its distance from the origin. But the transformation formulae shew that each particle describes the hodograph of the other. Remembering that the hodograph of a conic described under a central force is itself a conic it is thus proved that *if a conic be described under the central force* $r\mathrm{F}(x, y, \dot{x}, \dot{y})$, *a conic will also be described under the force* $r/\mathrm{F}(\dot{x}, \dot{y}, x, y)$. (We shall hereafter refer to this as Suchar's Theorem.)

4. The method employed by Suchar in obtaining the laws (*i*) to (*vi*) is rather tentative. The object of the present paper is to give a more general and rigorous, though less simple, method by which I have been able to obtain all the laws excepting (*i*) and (*ii*) without using Suchar's Theorem, and by which, I believe, the first two laws can also

[1] Bull. de la Soc. des Sciences, t XXXIII, Comptes Rendus, Vol. 135, p. 679.

be obtained. These two laws can however be easily obtained from the well-known laws (*vii*) and (*viii*) by the application of Suchar's theorem.

5. Let (x_1, y_1) be the co-ordinates and r_1 the distance from the origin of a particle moving under the central force F_1, at the time t_1. Its equations of motion are

$$\frac{d^2x_1}{dt_1^2}=F_1\frac{x_1}{r_1}, \qquad \frac{d^2y_1}{dt_1^2}=F_1\frac{y_1}{r_1}$$

whence

$$x_1\frac{dy_1}{dt_1}-y_1\frac{dx_1}{dt_1}=\text{a constant}=h(\text{say})$$

Making the homographic transformation

$$\frac{x_1}{y_1}=x, \quad y_1=\frac{1}{y}, \quad \frac{dt_1}{y_1^2}=dt,$$

we have

$$\frac{dx}{dt}=-h, \ \frac{dy}{dt}=-\frac{dy_1}{dt_1}$$

whence

$$\frac{d^2x}{dt^2}=0, \quad \frac{d^2y}{dt^2}=-\frac{d^2y_1}{dt_1^2}\cdot\frac{dt_1}{dt}=-F_1\frac{y_1^3}{r_1}.$$

Thus the equations of motion of a particle, whose co-ordinates are (x,y) at the time t, are

$$\frac{d^2x}{dt^2}=0, \quad \frac{d^2y}{dt^2}=Y \qquad \cdots \quad (5\cdot1)$$

where

$$Y=-F_1\frac{y_1^3}{r_1} \qquad \cdots \quad (5\cdot2)$$

From the nature of the transformation it follows that if the curve described by the first particle be a conic, that described by the second will also be a conic. The determination of F_1 thus reduces to the determination of a force Y acting parrallel to a fixed direction under which a particle can describe a conic.

6. Since the equations of motion of (x,y) are $(5\cdot1)$, the differential equation of its trajectory is

$$\frac{d^2y}{dx^2}=\frac{Y}{h^2},$$

where $$\frac{dx}{dt}=-h.$$

Now the differential equation of all conics is

$$\frac{d^3}{dx^3}\left[\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}\right]=0.$$

Thus Y must satisfy the equation

$$\frac{d^3}{dx^3}\left[(\mathrm{Y})^{-\frac{2}{3}}\right]=0.$$

Since the initial conditions are perfectly arbitrary, Y must be such as to satisfy this equation for all values of x, y, h and $\frac{dy}{dx}$.

7. Let us assume Y to be given by

$$\mathrm{Y}=[\phi(x,y,y')]^{-\frac{3}{2}} \qquad \ldots \quad (7\cdot1)$$

where y' stands for $\frac{dy}{dx}$.

Then ϕ must satisfy the equation

$$\frac{d^3}{dx^3}[\phi(x,y,y')]=0.$$

Performing the differentiation and remembering that

$$\frac{d^2y}{dx^2}=\frac{\mathrm{Y}}{h^2}=\frac{\phi^{-\frac{3}{2}}}{h^2},$$

the above equation becomes

$$\frac{\partial^3\phi}{\partial x^3}+3y'\frac{\partial^3\phi}{\partial x^2\partial y}+3y'^2\frac{\partial^3\phi}{\partial x\partial y^2}+y'^3\frac{\partial^3\phi}{\partial y^3}$$

$$+\frac{3\phi^{-\frac{7}{2}}}{4h^2}\left[4\phi^2\left\{\frac{\partial^3\phi}{\partial x^2\partial y'}+\frac{\partial^2\phi}{\partial x\partial y}+y'\left(2\frac{\partial^3\phi}{\partial x\partial y\partial y'}+\frac{\partial^2\phi}{\partial y^2}\right)\right.\right.$$

$$\left.+y'^2\frac{\partial^3\phi}{\partial y^2\partial y'}\right\}-2\phi\left\{\frac{\partial\phi}{\partial x}\cdot\frac{\partial\phi}{\partial y}+3\frac{\partial\phi}{\partial x}\cdot\frac{\partial}{\partial x\partial y'}\right.$$

$$\left.+\frac{\partial^2\phi}{\partial x^2}\cdot\frac{\partial\phi}{\partial y'}\right\}-2y'\phi\left\{\left(\frac{\partial\phi}{\partial y}\right)^2+3\frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\phi}{\partial x\partial y'}+3\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial y\partial y'}\right.$$

$$+2\frac{\partial\phi}{\partial y'}\cdot\frac{\partial^2\phi}{\partial x\partial y}\Big\}+5\frac{\partial\phi}{\partial y'}\Big\{\Big(\frac{\partial\phi}{\partial x}\Big)^2+2y'\frac{\partial\phi}{\partial x}\cdot\frac{\partial\phi}{\partial y}\Big\}$$

$$+y'^2\Big\{5\frac{\partial\phi}{\partial y'}\Big(\frac{\partial\phi}{\partial y}\Big)^2-2\phi\Big(3\frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\phi}{\partial y\partial y'}+\frac{\partial\phi}{\partial y'}\cdot\frac{\partial^2\phi}{\partial y^2}\Big)\Big\}\Big]$$

$$+\frac{3\phi^{-5}}{4h^4}\Big[4\phi^2\Big(\frac{\partial^3\phi}{\partial x\partial y'^2}+\frac{\partial^2\phi}{\partial y\partial y'}\Big)-2\phi\Big(2\frac{\partial\phi}{\partial y}\cdot\frac{\partial\phi}{\partial y'}$$

$$+5\frac{\partial\phi}{\partial y'}\cdot\frac{\partial^2\phi}{\partial x\partial y'}+3\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial y'^2}\Big)+13\Big(\frac{\partial\phi}{\partial y'}\Big)^2\Big(\frac{\partial\phi}{\partial x}+y'\frac{\partial\phi}{\partial y}\Big)$$

$$+y'\Big\{4\phi^2\frac{\partial^3\phi}{\partial y\partial y'^2}-2\phi\Big(5\frac{\partial\phi}{\partial y'}\cdot\frac{\partial^2\phi}{\partial y\partial y'}+3\frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\phi}{\partial y'^2}\Big)\Big\}\Big]$$

$$+\frac{\phi^{-\frac{13}{2}}}{h^6}\Big[\phi^2\frac{\partial^3\phi}{\partial y'^3}-6\phi\frac{\partial\phi}{\partial y'}\cdot\frac{\partial^2\phi}{\partial y'^2}+6\Big(\frac{\partial\phi}{\partial y'}\Big)^3\Big]=0.$$

8. Since the above equation is to be true for all values of h, ϕ must satisfy the following four differential equations:

$$\frac{\partial^3\phi}{\partial x^3}+3y'\frac{\partial^3\phi}{\partial x^2\partial y}+3y'^2\frac{\partial^3\phi}{\partial x\partial y^2}+y'^3\frac{\partial^3\phi}{\partial y^3}=0 \quad \dots \quad \text{(I)}$$

$$4\phi^2\Big\{\frac{\partial^3\phi}{\partial x^2\partial y'}+\frac{\partial^2\phi}{\partial x\partial y}+y'\Big(2\frac{\partial^3\phi}{\partial x\partial y\partial y'}+\frac{\partial^2\phi}{\partial y^2}\Big)+y'^2\frac{\partial^3\phi}{\partial y^2\partial y'}\Big\}$$

$$-2\phi\Big[\frac{\partial\phi}{\partial x}\cdot\frac{\partial\phi}{\partial y}+3\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial x\partial y'}+\frac{\partial^2\phi}{\partial x^2}\cdot\frac{\partial\phi}{\partial y'}$$

$$+y'\Big\{\Big(\frac{\partial\phi}{\partial y}\Big)^2+3\frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\phi}{\partial x\partial y'}+3\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial y\partial y'}$$

$$+2\frac{\partial\phi}{\partial y'}\cdot\frac{\partial^2\phi}{\partial x\partial y}\Big\}\Big]+5\frac{\partial\phi}{\partial y'}\Big\{\Big(\frac{\partial\phi}{\partial x}\Big)^2+2y'\frac{\partial\phi}{\partial x}\cdot\frac{\partial\phi}{\partial y}\Big\}$$

$$+y'^2\Big\{5\frac{\partial\phi}{\partial y'}\Big(\frac{\partial\phi}{\partial y}\Big)^2-2\phi\Big(3\frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\phi}{\partial y\partial y'}$$

$$+\frac{\partial\phi}{\partial y'}\cdot\frac{\partial^2\phi}{\partial y^2}\Big)\Big\}=0. \quad \dots \quad \text{(II)}$$

$$4\phi^2\left(\frac{\partial^3\phi}{\partial x\partial y'^2}+\frac{\partial^2\phi}{\partial y\partial y'}\right)-2\phi\left(2\frac{\partial\phi}{\partial y}\cdot\frac{\partial\phi}{\partial y'}+5\frac{\partial\phi}{\partial y'}\cdot\frac{\partial^2\phi}{\partial x\partial y'}\right.$$

$$\left.+3\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial y'^2}\right)+y'\left\{4\phi^2\frac{\partial^3\phi}{\partial y\partial y'^2}-2\phi\left(5\frac{\partial\phi}{\partial y'}\cdot\frac{\partial^2\phi}{\partial y\partial y'}\right.\right.$$

$$\left.\left.+3\frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\phi}{\partial y'^2}\right)\right\}+13\left(\frac{\partial\phi}{\partial y'}\right)^2\left(\frac{\partial\phi}{\partial x}+y'\frac{\partial\phi}{\partial y}\right)=0. \quad \ldots \quad \text{(III)}$$

$$\phi^2\frac{\partial^3\phi}{\partial y'^3}-6\phi\frac{\partial\phi}{\partial y'}\cdot\frac{\partial^2\phi}{\partial y'^2}+6\left(\frac{\partial\phi}{\partial y'}\right)^3=0. \qquad \ldots \quad \text{(IV)}$$

9. In order to solve these four equations simultaneously, consider at first the equation (IV).

Putting $\phi=\frac{1}{z}$, this reduces to

$$\frac{d^3z}{dy'^3}=0.$$

Hence
$$z=\frac{1}{\phi}=f_1(x,y)y'^2+f_2(x,y)y'+f_3(x,y) \qquad \ldots \quad (9\cdot1)$$

where f_1, f_2 and f_3 are some unknown functions to be determined from the equations (I), (II) and (III).

The laws (VII) and (VIII) can be easily deduced, as has already been shewn by Appell,[1] by assuming ϕ to be independent of y' in which case the differential equations become very much simplified and can be completely solved. In the present case the general solution of the equations seems very difficult and we shall try to obtain some particular solutions only.

10. Let us assume in the first place that

$$f_2=0,\ f_3=0 \text{ and } f_1=\frac{1}{f(x,y)}, \text{ so that } \phi=\frac{f(x,y)}{y'^2}.$$

Substituting this value in the equations (I), (II) and (III) and remembering that they must be satisfied for all values of y', we obtain the following six equations

$$\frac{\partial^3 f}{\partial x^3}=0,\quad \frac{\partial^3 f}{\partial x^2\partial y}=0,\quad \frac{\partial^3 f}{\partial x\partial y^2}=0,\quad \frac{\partial^3 f}{\partial y^3}=0,$$

$$2f\frac{\partial^2 f}{\partial x^2}-\left(\frac{\partial f}{\partial x}\right)^2=0,\quad 2f\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y}=0.$$

[1] loc. cit.

The first four of these equations shew that f must be at most a quadratic function of x and y, and the last two equations shew that f must have one of the forms

$$f=(ax+by+c)^2$$

or

$$f=(ay^2+2by+c).$$

Since

$$y'=\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{1}{h}\,\frac{dy_1}{dt_1}=\frac{1}{h}\dot{y}_1,$$

and

$$x=\frac{x_1}{y_1}\qquad y=\frac{1}{y_1},$$

we have

$$\phi=\frac{(ax+by+c)^2}{y'^2}=\frac{h^2(ax_1+cy_1+b)^2}{y_1^2\dot{y}_1^2}$$

or

$$\phi=\frac{ay^2+2by+c}{y'^2}=\frac{h^2(cy^2+2by_1+a)}{y_1^2\dot{y}_1^2}.$$

Hence, from (5·2) and (7·1),

$$F_1=h^2\dot{y}_1^3(ax_1+cy_1+b)^{-3}r_1$$

or

$$F_1=h^2\dot{y}_1^3(cy_1^2+2by_1+a)^{-\frac{3}{2}}r_1.$$

These are evidently the forms (*iii*) and (*iv*) of Suchar.

11. Let us now attempt to find a more general solution. Substituting the value of ϕ given by (9·1) in the equation (I), we obtain, on equating to zero the coefficient of the highest power of y' and the term independent of y', the following two equations

$$6\left(\frac{\partial f_1}{\partial y}\right)^3-6f_1\frac{\partial f_1}{\partial y}\cdot\frac{\partial^2 f_1}{\partial y^2}+f_1^2\frac{\partial^3 f_1}{\partial y^3}=0 \qquad \dots \quad (11·1)$$

$$6\left(\frac{\partial f_3}{\partial x}\right)^3-6f_3\frac{\partial f_3}{\partial x}\cdot\frac{\partial^2 f_3}{\partial x^2}+f_3^2\frac{\partial^3 f_3}{\partial x^3}=0 \qquad \dots \quad (11·2)$$

These equations being of the same form as (IV), their general solutions are

$$f_1(x,y)=\{\theta_1(x)y^2+\theta_2(x)y+\theta_3(x)\}^{-1} \qquad \dots \quad (11·3)$$

$$f_3(x,y)=\{\psi_1(y)x^2+\psi_2(y)x+\psi_3(y)\}^{-1} \qquad \dots \quad (11·4)$$

where θ_1, θ_2, θ_3, ψ_1, ψ_2 and ψ_3 are unknown functions to be determined from the remaining equations.

As the adoption of these general solutions is bound to lead to complications, we shall assume for the present that f_1 is a function of x only and f_3 that of y only, which is tantamount to the assumption

$$\theta_1=\theta_2=\psi_1=\psi_2=0.$$

Making this assumption and equating to zero the coefficients of the other powers of y', we obtain the following eight equations

$$\frac{\partial^3 f_2}{\partial x^3}=0 \qquad \ldots \quad (11{\cdot}41)$$

$$\frac{\partial^3 f_2}{\partial y^3}=0 \qquad \ldots \quad (11{\cdot}42)$$

$$6\,\frac{\partial^2 f_2}{\partial x^2}\left(\frac{\partial f_2}{\partial x}+\frac{\partial f_3}{\partial y}\right)-f_3\left(\frac{\partial^3 f_1}{\partial x^3}+3\,\frac{\partial^3 f_2}{\partial x^2\,\partial y}\right)=0 \qquad \ldots \quad (11{\cdot}43)$$

$$6\,\frac{\partial^2 f_2}{\partial y^2}\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}\right)-f_1\left(\frac{\partial^3 f_3}{\partial y^3}+3\,\frac{\partial^3 f_2}{\partial x\,\partial y^2}\right)=0 \qquad \ldots \quad (11{\cdot}44)$$

$$6\left(\frac{\partial f_2}{\partial x}+\frac{\partial f_3}{\partial y}\right)\left[f_2\,\frac{\partial^2 f_2}{\partial x^2}+f_3\left(\frac{\partial^2 f_1}{\partial x^2}+2\,\frac{\partial^2 f_2}{\partial x\,\partial y}+\frac{\partial^2 f_3}{\partial y^2}\right)\right.$$

$$\left.-\left(\frac{\partial f_2}{\partial x}+\frac{\partial f_3}{\partial y}\right)^2\right]+6f_3\,\frac{\partial^2 f_2}{\partial x^2}\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}\right)$$

$$-2f_2f_3\left(\frac{\partial^3 f_1}{\partial x^3}+3\,\frac{\partial^3 f_2}{\partial x^2\,\partial y}\right)-f_3^2\left(\frac{\partial^3 f_3}{\partial y^3}+3\,\frac{\partial^3 f_2}{\partial x\,\partial y^2}\right)=0 \quad (11{\cdot}45)$$

$$6\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}\right)\left[f_2\,\frac{\partial^2 f_2}{\partial y^2}+f_1\left(\frac{\partial^2 f_1}{\partial x^2}+2\,\frac{\partial^2 f_2}{\partial x\,\partial y}+\frac{\partial^2 f_3}{\partial y^2}\right)\right.$$

$$\left.-\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}\right)^2\right]+6f_1\,\frac{\partial^2 f_2}{\partial y^2}\left(\frac{\partial f_2}{\partial x}+\frac{\partial f_3}{\partial y}\right)$$

$$-2f_1f_2\left(\frac{\partial^3 f_3}{\partial y^3}+3\,\frac{\partial^3 f_2}{\partial x\,\partial y^2}\right)-f_1^2\left(\frac{\partial^3 f_1}{\partial x^3}+3\,\frac{\partial^3 f_2}{\partial x^2\,\partial y}\right)=0 \quad (11{\cdot}46)$$

$$6\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}\right)\left[f_2\,\frac{\partial^2 f_2}{\partial x^2}+f_3\left(\frac{\partial^2 f_1}{\partial x^2}+2\,\frac{\partial^2 f_2}{\partial x\,\partial y}+\frac{\partial^2 f_3}{\partial y^2}\right)\right.$$

$$\left.-3\left(\frac{\partial f_2}{\partial x}+\frac{\partial f_3}{\partial y}\right)^2\right]+6\left(\frac{\partial f_2}{\partial x}+\frac{\partial f_2}{\partial y}\right)\left[f_1\,\frac{\partial^2 f_2}{\partial x^2}\right.$$

$$+f_2\left(\frac{\partial^2 f_1}{\partial x^2}+2\frac{\partial^2 f_2}{\partial x\partial y}+\frac{\partial^2 f_3}{\partial y^2}\right)+f_3\frac{\partial^2 f_2}{\partial y^2}\Big]$$

$$-2f_2f_3\left(\frac{\partial^3 f_3}{\partial y^3}+3\frac{\partial^3 f_2}{\partial x\partial y^2}\right)$$

$$-(f_2^2+2f_1f_3)\left(\frac{\partial^3 f_1}{\partial x^3}+3\frac{\partial^3 f_2}{\partial x^2\partial y}\right)=0 \quad \ldots \quad (11\cdot 47)$$

$$6\left(\frac{\partial f_2}{\partial x}+\frac{\partial f_3}{\partial y}\right)\left[f_3\frac{\partial^2 f_2}{\partial y^2}+f_1\left(\frac{\partial^2 f_1}{\partial x^2}+2\frac{\partial^2 f_2}{\partial x\partial y}+\frac{\partial^2 f_3}{\partial y^2}\right)\right.$$

$$\left.-3\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}\right)^2\right]+6\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}\right)\left[f_1\frac{\partial^2 f_2}{\partial x^2}+f_2\left(\frac{\partial^2 f_1}{\partial x^2}\right.\right.$$

$$\left.\left.+2\frac{\partial^2 f_2}{\partial x\partial y}+\frac{\partial^2 f_3}{\partial y^2}\right)+f_3\frac{\partial^2 f_2}{\partial y^2}\right]-2f_1f_2\left(\frac{\partial^3 f_1}{\partial x^3}+3\frac{\partial^3 f_2}{\partial x^2\partial y}\right)$$

$$-(f_2^2+2f_1f_3)\left(\frac{\partial^3 f_3}{\partial y^3}+3\frac{\partial^3 f_2}{\partial x\partial y^2}\right) \quad \ldots \quad (11\cdot 48)$$

Remembering that f_1 is a function of x only and f_3 a function of y only, it is clear that the equations (11·43) to (11·48) are all satisfied by the two equations

$$\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}=0$$

and

$$\frac{\partial f_2}{\partial x}+\frac{\partial f_3}{\partial y}=0.$$

These equations give

$$\frac{d^2 f_1}{dx^2}=-\frac{\partial^2 f_2}{\partial x\partial y}=\frac{d^2 f_3}{dy^2}$$

Since $\frac{d^2 f_1}{dx^2}$ cannot contain y and $\frac{d^2 f_3}{dy^2}$ cannot contain x, each of these expressions must be a constant.

Hence $$\frac{d^3 f_1}{dx^3}=0,\quad \frac{d^3 f_3}{dy^3}=0$$

$$\frac{\partial^3 f_2}{\partial x^2\partial y}=0,\quad \frac{\partial^3 f_2}{\partial x\partial y^2}=0.$$

It follows that f_1, f_2 and f_3 must be of the forms

$$f_1=ax^2+2bx+d$$

$$f_2=-2(axy+by+cx+e)$$

$$f_3=ay^2+2cy+f$$

or
$$af_1=(ax+b)^2+a'$$
$$af_2=-2(ax+b)(ay+c)+2b'$$
$$af_3=(ay+c)^2+c'$$
where
$$a'=d-b^2,\ b'=bc-e,\ c'=f-c^2.$$

12. Let us now put $a'=b'=c'=0$.

Then ϕ becomes $a\{(ax+b)y'-(ay+c)\}^{-2}$.

It may be easily verified that this value of ϕ satisfies the equations (II) and (III).

Thus we get from (5·2) and (7·1),
$$F_1=ay_1^{-3}\{(ax+b)y'-(ay+c)\}^3 r_1$$
$$=\frac{a}{h}\ y_1^{-3}(a\dot{x}_1+b\dot{y}_1+c_1)^3 r_1$$
where
$$c_1=-ch.$$

This value of F_1 is obviously of the form (v) of Suchar.

13. Let us next put $a=b=c=0$, so that ϕ becomes
$$a(a'y'^2+2b'y'+c')^{-1}.$$

This expression for ϕ is also found to satisfy the equations (II) and (III).

In this case we get,
$$F_1=ay_1^{-3}(a'y'^2+2b'y'+c')^{\frac{3}{2}}\ r_1$$
$$=\frac{a}{h^3}y_1^{-3}\ (a'\dot{y}_1^2+2b_1\dot{y}_1+c_2)^{\frac{3}{2}}\ r_1,$$
where
$$b_1=b'h,\quad c_2=c'h^2.$$
Here F_1 is of the form (vi) of Suchar.

14. We have thus deduced the laws (iii) to (vi) directly from the differential equations (I) to (IV) by considering some particular solutions only, and it has been pointed out that the laws (vii) and ($viii$) can be obtained from these equations by assuming ϕ to be independent of y. It will be my endeavour to discuss the equations more fully in a subsequent paper in which I hope to be able to give the complete analytical solution of the problem.

My attention was drawn to Suchar's paper by Prof. G. Prasad.

On Circular Vortex rings of finite section in incompressible fluids.

BY

Nripendranath Sen, M.Sc.

The steady motion of a circular vortex ring in an incompressible fluid has been investigated by many eminent mathematicians including Helmholtz,[1] Kelvin,[2] Lewis,[3] Thomson,[4] Basset,[5] Hicks,[6] Chree,[7] and Dyson.[8] But a complete solution of the problem has not been obtained up till now. Assuming the cross-section of the ring to be circular and very small in comparison with its aperture so that by Maxwell's electrical analogy the whole matter may be supposed te be condensed on the circular axis, Lewis, Thomson, Basset and Chree have found the velocity of a ring of constant vorticity to be

$$\frac{k}{4\pi c}\left(log\frac{8c}{a}-1\right)$$

where k=strength of the whole vortex, c=radius of the circular axis, and a=radius of the cross-section. In a note to Prof. Tait's translation of Helmholtz's paper, Lord Kelvin gives the velocity to be

$$\frac{k}{4\pi c}\left(log\frac{8c}{a}-\frac{1}{4}\right)$$

This latter result was also given by Basset, Hicks and Dyson for a ring whose vorticity varies directly as the distance from the axis of

[1] Helmholtz—" Ueber Integrale der hydrodynamischen Gleichungen Welche den Wirbelbewegungen entsprechen." Crelle, Vol. 55, 1858.

[2] Kelvin—" Collected Scientific papers" Vol. 4, p. 67.

[3] Lewis—" Quart. Jour. Math." Vol. 16, 338-47, 1879.

[4] J. J. Thomson—" Motion of Vortex rings."

[5] Basset—Hydrodynamics, Part II.

[6] Hicks—" Phil. Trans. A Vol. 175 I, 1884. Also Vol. 176 II, 1885.

[7] Chree—" Proc. Edin. Math. Soc." Vol. 6, 1888.

[8] Dyson—" Potential of an anchor ring " Part I and II, Phil. Trans. A Vol. 184, 1893.

Also Gray—" Notes on Hydrodynamics" Phil. Mag. (6) Vol. 28, 13, 1914.

Lamb " Hydrodynamics " Ed. IV, 1916.

the ring. This discrepancy was tried to be explained by Dr. Chree who pointed out that the hypothesis of constant vorticity is not consistent with that of a truly circular section in which case he proved that the vorticity must vary inversely as the square of the distance from the axis of the ring, but he has not investigated the motion of vortex rings of finite section either of constant vorticity or of vorticity varying according to the law found out by him.

In the present paper, first the motion of vortex rings of finite section and constant vorticity has been solved and it has been shewn that the cross-section does not remain circular but is given by an equation of the form

$$r=a\left[1-\frac{36\lambda+25}{96}\frac{a^2}{c^2}\cos 2\theta-\frac{360\lambda+155}{3072}\frac{a^3}{c^3}\cos 3\theta\ldots\right]$$

where $\lambda=\log\frac{8c}{a}-2$.

As a first approximation the velocity of translation of a ring of constant vorticity and circular cross-section has been found to be

$$\frac{k}{4\pi c}\left[\log\frac{8c}{a}-\frac{3}{4}\right]$$

The slight difference in this result and that obtained by Thomson, Lewis and Chree has been explained at length in Art. 7 where it has been shewn that my result is perfectly accurate to the order of approximation adopted.

Secondly, the cross-section, velocity of translation and fluted oscillations of a vortex ring of finite section and vorticity varying as any power of the distance from the axis of the ring have been investigated, from which I have shewn that it is only when the vorticity varies inversely as the square root of the distance that the velocity to a first approximation is given by

$$\frac{k}{4\pi c}\left(\log\frac{8c}{a}-1\right).$$

When the vorticity obeys Chree's law referred to above the corresponding results have also been deduced from our general results; it has been shewn that even in this case the cross section does not remain circular but gets elongated in the direction of motion and the velocit to a first approximation has been found to be

$$\frac{k}{4\pi c}\left(\log\frac{8c}{a}-\frac{1}{4}\right).$$

I have also shewn how the results of Hicks and Dyson are easily deducible as a particular case of the general problem here studied.

My best thanks are due to Dr. Bibhutibhusan Datta for the great interest he has taken in the preparation of this paper.

PRELIMINARY REMARKS AND DEFINITIONS.

2. Let the "circular axis" of the vortex ring be defined to be the circle passing through the centre of area of the cross sections of the vortex filament and let the perpendicular to the plane of the circular axis through its centre be called the "axis" of the vortex. Also let

$2\,\omega$=vorticity, k=strength of the vortex

c=radius of the circular axis

ρ, ϕ, z=cylindrical co-ordinates of any point referred to the centre of the circular axis as origin and the axis of the ring as z-axis.

r=distance of any point from the circular axis

θ=inclination of this distance to the plane of the circular axis so that $\rho=c-r\cos\theta$.

V=velocity of the ring parallel to z—axis

$$J=\int_0^\pi \frac{c\cos\phi\, d\phi}{\{z'^2+c^2-2c\rho'\cos\phi+\rho'^2\}^{\frac{1}{2}}}$$

a=mean radius of the cross-section,

$$l=\log\frac{8c}{r}-2,\ s=\frac{r}{c}$$

$$\lambda=\log\frac{8c}{a}-2,\ \sigma=\frac{a}{c}$$

$$\nabla^2=\frac{d^2}{dc^2}+\frac{d^2}{dz'^2},\ \frac{d}{dc}=\nabla\cos a,\quad \frac{d}{dz'}=\nabla\sin a$$

ψ=stokes's stream function,

then, it is well-known that

$$\frac{\partial^2\psi}{\partial z^2}+\frac{\partial^2\psi}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}=0 \text{ outside the vortex filament} \quad \dots \quad (1)$$

and

$$\frac{\partial^2\psi}{\partial z^2}+\frac{\partial^2\psi}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}=-2\omega\rho \text{ insida the filament} \qquad \dots \quad (2)$$

The value of ψ at any point (ρ', ϕ', z') outside the filament may easily be seen to be given by

$$\psi=\frac{\rho'}{2\pi}\iiint\frac{\omega\,\rho\cos\phi\,d\rho\,dz\,d\phi}{\{(z'-z)^2+\rho'^2-2\rho\rho'\cos\phi+\rho^2\}^{\frac{1}{2}}} \qquad \dots \quad (3)^1$$

where the integral is to be taken throughout the volume of the vortex filament

CASE I. ω=CONSTANT.

3. It has been pointed out by Dr. C. Chree[2] that the hypothesis of constant vorticity is not consistent with the assumption that the vortex ring has a truly circular section, but he has not discussed what the cross-section should be if the vorticity be constant. Let us take up the case here.

Let the (r, θ) equation of the cross-section of the filament be

$$r=a\,(1+A_1\cos\theta+A_2\cos 2\theta+A_3\cos 3\theta+\dots)\dots$$

From the definition of the circular axis, $\displaystyle\int_0^{2\pi}\!\!\int_0^{r} r^2\cos\theta\,dr\,d\theta=0.$

i.e. $$A_1+A_2\ A_3+\dots=0$$

We shall presently prove that A_2 and A_3 are of the order σ^2 and σ^3 respectively.[3] A_1 is, therefore, of the order σ^5. Hence, neglecting σ^4 and higher powers of σ, the cross-section of the filament is given by

$$r=a\,(1+A_2\cos 2\theta+A_3\cos 3\theta) \qquad \dots \quad (4)$$

Putting x for $(r\cos\theta)$ in (3), we have

$$\psi=\frac{\omega\rho'}{2\pi}\iiint\frac{(c-x)\,d\rho\,dz\,d\phi}{\{(z'-z)^2+\rho'^2-2\rho'(c-x)\cos\phi+(c-x)^2\}^{\frac{1}{2}}}$$

[1] Basset—Hydrodynamics t II, p. 60.

[2] "Vortex rings in a compressible fluid" *ibid.*, p. 62.

[3] See results (28) and (29).

taken throughout the volume of the ring

$$=\frac{\omega\rho'}{\pi}\iint e^{-x\frac{d}{dc}-z\frac{d}{dz'}}\,dx\,dz\int_0^{\pi}\frac{c\cos\phi\,d\phi}{\{z'^2+\rho'^2-2\rho'c\cos\phi+c^2\}^{\frac{1}{2}}}$$

where the first integral is to be taken over the cross-section given by (4)

$$=\frac{\omega\rho'}{\pi}\iint e^{-x\frac{d}{dc}-z\frac{d}{dz'}}\,dx\,dz\,(\mathrm{J})\quad\ldots\quad\ldots\quad\ldots\quad(5)$$

Now, $\iint e^{-x\frac{d}{dc}-z\frac{d}{dz'}}\,dx\,dz$ taken over the cross-section (4)

$$=\int_0^{R}\int_0^{2\pi} e^{-r\nabla\cos(\theta-a)}\,r\,dr\,d\theta \text{ where } R=a\,(1+A_2\cos 2\theta+A_3\cos 3\theta)$$

$$=\int_0^{a}\int_0^{2\pi} e^{-r\nabla\cos(\theta-a)}\,r\,dr\,d\theta+\int_a^{R}\int_0^{2\pi} e^{-r\nabla\cos(\theta-a)}\,r\,dr\,d\theta$$

But, $\int_0^{a}\int_0^{2\pi} e^{-r\nabla\cos(\theta-a)}\,r\,dr\,d\theta$

$$=\int_0^{a}\int_0^{2\pi}\{I_0(r\nabla)-2I_1(r\nabla)\cos(\theta-a)+2I_2(r\nabla)\cos 2(\theta-a)-\text{etc.}\}\,r\,dr\,d\theta$$

where I_n is Bessel's function[1] of the nth order with imaginary modulus

$$=2\pi\int_0^{a} I_0(r\nabla)\,r\,dr=2\pi\frac{I_1(a\nabla)}{\nabla}a$$

$$=\pi a^2\left(1+\frac{a^2\nabla^2}{8}+\frac{a^4\nabla^4}{192}+\frac{a^6\nabla^6}{9216}+\frac{a^8\nabla^8}{73280}+\text{etc.}\right)$$

[1] Whittaker—"Modern Analysis" 17·7 and 17·1 Ex. 2.

Also, $\displaystyle\int_a^R\int_0^{2\pi} e^{-r\nabla\cos(\theta-\alpha)}\, r\,dr\,d\theta$

$2\pi a\,(A_2\cos 2\theta+A_3\cos 3\theta)$

$$=\int_0\int_0 e^{-(r+a)\nabla\cos(\theta-\alpha)}(a+r)\,dr\,d\theta$$

$2\pi a\,(A_2\cos 2\theta+A_3\cos 3\theta)$

$$=\int_0\int_0 e^{-a\nabla\cos(\theta-\alpha)}\{1-r\nabla\cos(\theta-\alpha)+\ldots\}\,(a+r)\,dr\,d\theta$$

$$=a^2\int_0^{2\pi} e^{-a\nabla\cos(\theta-\alpha)}(A_2\cos 2\theta+A_3\cos 3\theta)\,d\theta$$

neglecting σ^4 and higher powers of σ

$$=a^2\int_0^{2\pi}\{I_0(a\nabla)-2I_1(a\nabla)\cos(\theta-\alpha)+2I_2(a\nabla)\cos 2(\theta-\alpha)-2I_3(a\nabla)\cos 3(\theta-\alpha)+\text{etc.}\}$$

$$\times(A_2\cos 2\theta+A_3\cos 3\theta)\,d\theta$$

$$=2\pi a^2[A_2 I_2(a\nabla)\cos 2\alpha-A_3 I_3(a\nabla)\cos 3\alpha]$$

$$=\frac{\pi a^2}{4}\left\{A_2 a^2\nabla^2\cos 2\alpha\left(1+\frac{a^2\nabla^2}{12}+\frac{a^4\nabla^4}{384}+\ldots\right)-A_3\frac{a^3\nabla^3}{6}\left(1+\frac{a^2\nabla^2}{16}+\ldots\right)\right\}$$

$\therefore \displaystyle\iint e^{-x\frac{d}{dc}-z\frac{d}{d'z}}\,dx\,dz$ taken over the cross-section (4).

$$=\pi a^2\left\{1+\frac{a^2\nabla^2}{8}+\frac{a^4\nabla^4}{192}+\frac{a^6\nabla^6}{9216}+\text{etc.}\right.$$

$$+\frac{A_2}{4}\,a^2\,\nabla^2\cos 2a\left(1+\frac{a^2\,\nabla^2}{12}+\frac{a^4\,\nabla^4}{384}+\text{etc.}\right)$$

$$-\frac{A_3}{24}\,a^3\,\nabla^3\left(1+\frac{a^2\,\nabla^2}{16}+\text{etc.}\right)\cos 3a\Big\}\quad\ldots\quad\ldots\quad(6)^1$$

4. Sir F. W. Dyson has obtained by a different method an expression[1] for the above integral $\iint e^{-x\frac{d}{dc}-z\frac{d}{dz'}}\,dx\,dz$, which differs from that obtained by me in the co-efficient of A_3, he getting $\frac{A_3}{12}$ in the place of $\frac{A_3}{24}$ in (6). After carefully going through his calculations, I find that the difference is due to a mistake in calculation on his part, as by repeating his own process, I have got the expression (6).

5. From (5) and (6), at any point (ρ', ϕ', z') outside the filament,

$$\psi=\omega\,\rho'\,a^2\left[1+\frac{a^2\,\nabla^2}{8}+\frac{a^4\,\nabla^4}{192}+\frac{a^6\,\nabla^6}{9216}+\frac{a^8\,\nabla^8}{73280}+\text{etc.}\right.$$

$$+\frac{A_2\,a^2}{4}\left(2\frac{d^2}{dc^2}-\nabla^2\right)\left(1+\frac{a^2\,\nabla^2}{12}+\frac{a^4\,\nabla^4}{384}+\text{etc.}\right)$$

$$\left.-\frac{A_3\,a^3}{24}\left(4\frac{d^3}{dc^3}-3\,\nabla^2\frac{d}{dc}\right)\left(1+\frac{a^2\,\nabla^2}{16}+\text{etc.}\right)\right]J\quad\ldots\quad(7)$$

Now, it can be easily shewn that

$$\frac{d^2J}{dc^2}+\frac{d^2J}{dz'^2}-\frac{1}{c}\,\frac{dJ}{dc}=0\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad(8)^2$$

$$i.e.\ \nabla^2\,J=\left(\frac{1}{c}\,\frac{d}{dc}\right)$$

[1] *Loc. cit.* Part II, p. 1083.

[2] This is evident from (31). Since ψ in (31) must satisfy equation (1) *viz.* $\frac{\partial^2\psi}{\partial z'^2}+\frac{\partial^2\psi}{\partial\rho'^2}-\frac{1}{\rho'}\,\frac{\partial\psi}{\partial\rho'}=0$ it is easy to see by writing c for ρ' and ρ' for c both in this equation and the expression for ψ *viz.* $\frac{k\rho'J}{2\pi}$ given in (31) that $\frac{d^2J}{dc^2}+\frac{d^2J}{dz'^2}-\frac{1}{c}\,\frac{dJ}{dc}=0.$

$$\therefore \nabla^4 J = \nabla^2 \left(\frac{1}{c} \frac{d}{dc} \right) J = \frac{1}{c} \frac{d}{dc} \nabla^2 J - \frac{2}{c} \left(\frac{1}{c} \frac{d^2}{dc^2} - \frac{1}{c^2} \frac{d}{dc} \right) J$$

$$= \left(\frac{1}{c} \frac{d}{dc} \right)^2 J - 2 \left(\frac{1}{c} \frac{d}{dc} \right)^2 J = 1\,(-1) \left(\frac{1}{c} \frac{d}{dc} \right)^2 J$$

Similarly $\nabla^6 J = 1\,(-1)\,(-3) \left(\frac{1}{c} \frac{d}{dc} \right)^3 J$

... ...

$$\nabla^{2n} J = 1\,(-1)\,(-3)\ldots(3-2n) \left(\frac{1}{c} \frac{d}{dc} \right)^n J \quad \ldots \quad \ldots \quad (8A)$$

where n is any positive integer.

Also $$\frac{d^2}{dc^2} = c^2 \left(\frac{1}{c} \frac{d}{dc} \right)^2 + \left(\frac{1}{c} \frac{d}{dc} \right) \quad \ldots \quad \ldots \quad (9)$$

$$\frac{d^3}{dc^3} = c^3 \left(\frac{1}{c} \frac{d}{dc} \right)^3 + 3c \left(\frac{1}{c} \frac{d}{dc} \right)^2 \quad \ldots \quad \ldots \quad (9A)$$

Again, $$J \frac{\rho'}{c} = l - \frac{l+1}{2} s \cos\theta + \left(\frac{2l+5}{16} - \frac{l}{16} \cos 2\theta \right) s^2$$

$$+ \left(\frac{3l+5}{64} \cos\theta - \frac{3l-1}{192} \cos 3\theta \right) s^3 + \left(\frac{12\,l+11}{2048} \right.$$

$$\left. + \frac{12l+17}{768} \cos 2\theta - \frac{15l-8}{3072} \cos 4\theta \right) s^4 + \text{etc.} \quad \ldots \quad (10)$$

$$\frac{\rho'}{c} \left(\frac{1}{c} \frac{d}{dc} \right) J = \frac{1}{c^2 s} \left\{ -\cos\theta + \left(\frac{2l+3}{4} + \frac{\cos 2\theta}{4} \right) s \right.$$

$$+ \left(\frac{4l+1}{32} \cos\theta + \frac{\cos 3\theta}{32} \right) s^2 + \left(-\frac{4l+7}{128} + \frac{4l+1}{64} \cos 2\theta \right.$$

$$\left. \left. + \frac{\cos 4\theta}{128} \right) s^3 + \text{etc.} \right\} \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

$$\frac{\rho'}{c} \left(\frac{1}{c} \frac{d}{dc} \right)^2 J = \frac{1}{c^4 s^2} \left\{ \cos 2\theta - \frac{\cos\theta + \cos 3\theta}{4} s - \left(\frac{12l+9}{32} \right. \right.$$

$$\left. \left. + \frac{\cos 2\theta}{4} + \frac{\cos 4\theta}{32} \right) s^2 - \text{etc.} \right\} \quad \ldots \quad (12)$$

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)^3 J=-\frac{1}{c^6 s^3}\left\{2\cos 3\theta+\left(\cos 2\theta-\frac{\cos 4\theta}{2}\right)s+\ldots\right\} \quad (13)$$

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)^4 J=\frac{1}{c^8 s^4}\left\{6\cos 4\theta+\ldots\right\} \quad \ldots \quad \ldots \quad (14)$$

From (4), on the surface of the ring $s=\frac{r}{c}=\sigma\,(1+A_2\cos 2\theta$

$+A_3\cos 3\theta)$.

Hence, from the above relations (10) to (14), we have on the surface of the ring,

$$\frac{\rho'}{c}J=\lambda+\frac{2\lambda+5}{16}\sigma^2+\left(-\frac{\lambda+1}{2}+\frac{3\lambda+5}{64}\sigma^2-\frac{\lambda}{4}A_2\right)\sigma\cos\theta$$

$$-\left(\frac{\lambda\sigma^2}{16}+A_2\right)\cos 2\theta-\left(\frac{3\lambda-1}{192}\sigma^3+A_3+\frac{\lambda\sigma A_2}{4}\right)\cos 3\theta \quad (15)$$

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)J=\frac{1}{c^2\sigma}\left\{\frac{2\lambda+3}{4}\sigma-\frac{4\lambda+7}{128}\sigma^3+\left(-1+\frac{4\lambda+1}{32}\sigma^2\right.\right.$$

$$\left.+\frac{A_2}{2}\right)\cos\theta+\left(\frac{\sigma}{4}+\frac{4\lambda+1}{64}\sigma^3-\frac{\sigma}{2}A_2+\frac{A_3}{2}\right)\cos 2\theta$$

$$\left.+\left(\frac{\sigma^2}{32}+\frac{A_2}{2}\right)\cos 3\theta+\left(\frac{\sigma^3}{128}+\frac{A_3}{2}\right)\cos 4\theta\right\} \quad \ldots \quad (16)$$

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)^2 J=\frac{1}{c^4\sigma^2}\left\{-\frac{12\lambda+9}{32}\sigma^2-A_2-\frac{\sigma}{4}\cos\theta\right.$$

$$\left.+\cos 2\theta\left(1-\frac{\sigma^2}{4}\right)-\frac{\sigma}{4}\cos 3\theta-\left(\frac{\sigma^2}{32}+A_2\right)\cos 4\theta\right\} \quad (17)$$

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)^3 J=\frac{1}{c^6\sigma^3}\left\{-2\cos 3\theta-\sigma\cos 2\theta+\frac{\sigma}{2}\cos 4\theta\right\} \quad (18)$$

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)^4 J=\frac{6}{c^8\sigma^4}\cos 4\theta \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

The expressions (10) to (14) have been taken from Dyson's memoir[1]

From the above relations it is easy to see that

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)^n J \text{ is of the order } \frac{1}{c^{2n}\sigma^n}$$

[1] *Loc. cit.* Part I, p. 54, Part II, p. 1086-87.

Hence, $a^r c^{2n-r} \frac{\rho'}{c} \left(\frac{1}{c} \frac{d}{dc} \right)^n$ J is of the order σ^{r-n} ... (19A)

Also $k = \iint 2\, \omega\, r\, dr\, d\theta$ taken over the cross-section of the filament

$=2\pi\omega a^2$ neglecting σ^4 and higher powers of σ ... (20)

Hence, always neglecting s^4 and higher powers of s, we have from (7), after a little simplification with the help of above equations, that ψ at any point (r, θ) outside the filament and very near to it,

$$= \frac{k\rho'}{2\pi} \left[1 + \frac{a^4}{2} (A_2 + \tfrac{1}{2}) \left(\frac{1}{c} \frac{d}{dc} \right) + \left(-\frac{a^4}{192} + \frac{a^2 c^2}{2} A_2 \right) \left(\frac{1}{c} \frac{d}{dc} \right)^2 \right.$$

$$\left. + \left(\frac{a^6}{3072} - \frac{a^3 c^3}{6} A_3 + \frac{a^4 c^2}{24} A_2 \right) \left(\frac{1}{c} \frac{d}{dc} \right)^3 \right] J \qquad \ldots \quad (21)$$

Whence, by (15) to (19), we have on the surface of the ring,

$$\psi = \frac{kc}{2\pi} \left[\text{constant} - \left(\frac{4\lambda+5}{8} + \frac{4\lambda+5}{16} A_2 - \frac{3\lambda+4}{48} \sigma^2 \right) \sigma \cos\theta \right.$$

$$- \left(\frac{A_2}{2} + \frac{12\lambda-5}{192} \sigma^2 \right) \cos 2\theta - \left(\tfrac{2}{3} A_3 + \frac{12\lambda+7}{48} \sigma A_2 \right.$$

$$\left. \left. + \frac{8\lambda-5}{512} \sigma^3 \right) \cos 3\theta \right] \qquad \ldots \quad \ldots \quad \ldots \quad (22)$$

Since, the vortex ring moves with a velocity of translation V parallel to z-axis, we have

$\psi = \frac{V\rho^2}{2}$ + const on the surface of the ring

$$= \frac{Vc^2}{2} \left[\text{const.} - (2 + A_2)\, \sigma \cos\theta + \frac{\sigma^2}{2} \cos 2\theta - \sigma A_2 \cos 3\theta \right]$$

from (4), neglecting σ^4 ond higher powers of σ ... ... (23)

Since the relation (23) is to be true for all values of θ, we have fröm (22) and (23) by equating the co-efficients of $\cos\theta$, $\cos 2\theta$, $\cos 3\theta$ respectively.

$$Vc \left(1 + \frac{A_2}{2} \right) = \frac{k}{\pi} \left\{ \frac{4\lambda+5}{16} \left(1 + \frac{A_2}{2} \right) - \frac{3\lambda+4}{96} \sigma^2 \right\} \qquad \ldots \quad (24)$$

$$Vc\,\sigma^2 = -\frac{k}{\pi}\left(A_2 + \frac{12\lambda-5}{96}\,\sigma^2\right) \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

$$Vc\,\sigma\,A_2 = \frac{k}{\pi}\left(\tfrac{2}{3}\,A_3 + \frac{12\lambda+7}{48}\,\sigma\,A_2 + \frac{8\lambda-5}{512}\,\sigma^3\right) \quad \ldots \quad (26)$$

From (24), dividing both sides by $1+\frac{A_2}{2}$ and neglecting $A_2\,\sigma^2$, we have

$$V = \frac{k}{\pi c}\left(\frac{4\lambda+5}{16} - \frac{3\lambda+4}{96}\,\sigma^2\right) \quad \ldots \quad \ldots \quad \ldots \quad (27)$$

Substituting this value of V in (25), we have

$$A_2 = -\frac{36\lambda+25}{96}\,\sigma^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

Substituting the values of V and A_2 in (26), we obtain

$$A_3 = -\frac{360\lambda+155}{3072}\,\sigma^3 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

Hence, from (4), the (r,θ) equation of the cross-section is

$$r = a\left[1 - \frac{36\lambda+25}{96}\,\sigma^2\cos 2\theta - \frac{360\lambda+155}{3072}\,\sigma^3\cos 3\theta\right] \quad \ldots \quad (30)$$

From (30), it is evident that the ring does not remain circular but gets elongated in the direction of motion $\left(\theta=\frac{\pi}{2}\right)$ and that it may be regarded circular when and only when we neglect quantities of the order σ^2 and higher powers of σ.

6. In the previous Art. we have found out the values of V, A_2, A_3 correct to σ^3. The same method may be applied to find the velocity and shape of the cross-section correct to any power of σ, but the values of A_2, A_3 obtained in (28) and (29) give the shape of the cross-section to a fair degree of approximation even in the case of very thick rings.

Thus when $\sigma=\cdot3$; $V=\frac{k}{\pi c}\times\cdot626$; $A_2=-\cdot067$; $A_3=-\cdot005$

$\sigma=\cdot25$; $V=\frac{k}{\pi c}\times\cdot674$; $A_2=-\cdot051$; $A_3=-\cdot003$

$\sigma=\cdot2$; $V=\frac{k}{\pi c}\times\cdot733$; $A_2=-\cdot036$; $A_3=-\cdot002$

$\sigma=\cdot1$; $V=\frac{k}{\pi c}\times\cdot907$; $A_2=-\cdot012$; $A_3=-\cdot0003$

CRITICISM OF THE RESULTS OBTAINED BY SIR J. J. THOMSON AND OTHERS.

7. If the vortex ring be so thin that σ^2 and higher powers are negligible, we have found that the cross-section may be regarded as circular and in that case from (27) we have,

$$V=\frac{k}{16\pi c}(4\lambda+5)=\frac{k}{4\pi c}\left(\log\frac{8c}{a}-\tfrac{3}{4}\right)$$

But the expression for the velocity of translation of such small rings obtained by Thomson,[1] Lewis,[2] and Chree[3] is

$$V=\frac{k}{4\pi c}\left(\log\frac{8c}{a}-1\right)$$

Thus these two velocities differ by a quantity of the order $\frac{k}{c}$. Let us investigate into the cause of this difference.

Both Sir J. J. Thomson[4] and Dr. Chree[5] have, following the common practice in calculating the velocity, regarded the vortex filament as concentrated in the circular axis. We shall presently prove that this cannot be done without committing an error of the order $\frac{k}{c}$ in the expression for the velocity,

For, if it be supposed that the cross-section of the filament is circular and very small and if it be further supposed by Maxwell's electrical analogy that "the action of a vortex ring of this shape will be the same as one of equal strength condensed at the central line of the vortex core," the corresponding expression for ψ may be easily written down from (3) by putting

$z=0$, $\rho=c$, $2\omega\, d\rho\, dz=k=$strength of the vortex.

Hence, in the present case ψ at any point (ρ', ϕ', z') is given by

$$\psi=\frac{k\rho'}{2\pi}\int_0^{\pi}\frac{c\cos\phi\, d\phi}{\{z'^2+\rho'^2-2c\rho'\cos\phi+c^2\}^{\frac{1}{2}}}$$

$$=\frac{k\rho'J}{2\pi}=\frac{kc}{2\pi}\left[l-\frac{l+1}{2}s\cos\theta+\text{etc.}\right]\text{ from (10)} \quad \ldots \quad (31)^{6}$$

[1] *Loc. cit.* p. 33, Equation (41).

[2] *Loc. cit.* p. 338-47.

[3] *Loc. cit.* p. 61, Equation (14).

[4] Loc. *cit.* p. 13, Art. 8, para. I.

[5] *Loc. cit.* p. 59, para 2.

[6] Evidently this is true whether the vorticity be constant or variable.

If the ring has a velocity of translation V parallel to z—axis we have,

$$\psi=\frac{V\rho^2}{2}+\text{const}$$

on the surface of the ring $r=a$, *i.e.*,

$$\frac{k\rho' J}{2\pi}=\frac{V\rho^2}{2}+\text{const}$$

when $r=a$,

$\therefore$ From (31), putting $s=\sigma$, $l=\lambda$, we have,

$$\frac{kc}{2\pi}\left\{\lambda-\frac{\lambda+1}{2}\sigma\cos\theta+\ldots\right\}=\frac{Vc^2}{2}\left[\text{constant}-2\sigma\cos\theta+\frac{\sigma^2}{2}\cos 2\theta\right]$$

Neglecting σ^2 and higher powers of σ in the above equation, we have by equating the co-efficients of $\sigma\cos\theta$ in both sides,

$$V=\frac{k}{4\pi c}(\lambda+1)=\frac{k}{4\pi c}\left(\log\frac{8c}{a}-1\right)$$

which is the expression for the velocity obtained by Prof. J. J. Thomson, Dr. C. Chree, and others.

Now, the expression for ψ for a circular ring of very small circular cross-section may be easily deduced from our exact equation (21) by omitting A_2 and A_3, whence ψ for such a ring is given by

$$\psi=\frac{k\rho'}{2\pi}\left[1+\frac{a^2}{8}\left(\frac{1}{c}\frac{d}{dc}\right)-\frac{a^4}{192}\left(\frac{1}{c}\frac{d}{dc}\right)^2+\frac{a^6}{3072}\left(\frac{1}{c}\frac{d}{dc}\right)^3\right]J \quad (32)$$

If $\psi=\frac{k\rho' J}{2\pi}$ as given by (31) by following the method of treatment of Sir J. J. Thomson and others, evidently we reject 2nd, 3rd and other terms in the expression for ψ given by (32). But remembering that it is by equating the co-efficients of $\sigma\cos\theta$ in both sides of the equation $\psi=\frac{V\rho^2}{2}+\text{const.}$ on the surface of the ring, that the expression for the velocity is to obtained, it is absolutely necessary that all the terms in ψ given by (32), correct to first power of $\sigma\cos\theta$ must be retained. From (19 A),

$$\frac{\rho'}{c}a^2\left(\frac{1}{c}\frac{d}{dc}\right)J$$

is at the least of the order σ and must be retaind. Similarly,

$$\frac{\rho'}{c}a^4\left(\frac{1}{c}\frac{d}{dc}\right)^2 J$$

and

$$\frac{\rho'}{c}a^6\left(\frac{1}{c}\frac{d}{dc}\right)^3 J$$

are of the order σ^2 and σ^3 respectively and may be rejected. Hence the *correct* expression for ψ for a thin circular vortex ring is given by

$$\psi=\frac{k\rho'}{2\pi}\left[1+\frac{a^2}{8}\left(\frac{1}{c}\frac{d}{dc}\right)\right]J \qquad \ldots \quad (33)$$

$$=\frac{kc}{2\pi}\left\{l-\frac{4l+5}{8}s\cos\theta+\text{etc.}\right\} \text{ from (10) and (11)} \qquad \ldots \quad (34)$$

Since, $\psi=\frac{V\rho^2}{2}+\text{const}$ when $r=a$, we have from (34),

$$\frac{kc}{2\pi}\left\{\lambda-\frac{4\lambda+5}{8}\sigma\cos\theta+\text{etc.}\right\}=\frac{Vc^2}{2}\left[\text{const}-2\sigma\cos\theta+\ldots\right]$$

for all volues of θ.

Whence, equating the co-efficients of $\sigma\cos\theta$, we have the correct expression for the velocity given by

$$V=\frac{k}{16\pi c}\left(4\lambda+5\right)=\frac{k}{4\pi c}\left(\log\frac{8c}{a}-\frac{3}{4}\right)$$

which is the expression already obtained by us.[1]

CASE II. ω VARIABLE.

8. Dr. Chree[2] has proved that in a ring of circular cross-section, the vorticity is not constant but varies accordng to the law

$$\omega=\Omega\rho^{-2} \quad \text{where } \Omega=\text{constant.} \qquad \ldots \quad (35)^3$$

[1] Also See. Art 12, where it has been proved that $V=\frac{k}{4\pi c}\left(\log\frac{8c}{a}-1\right)$ only when vorticity $\infty\rho^{-\frac{1}{2}}$.

[2] Loc. cit., p. 62.

[3] See Art 11.

Let us make a more general assumption *viz.*,

$$\omega=\Omega\rho^n \text{ where } n \text{ is any quantity positive or negative} \qquad \ldots \quad (36)$$

Hence, from (3) ψ at any point (ρ', ϕ', z') outside the filament is given in this case by

$$\psi=\frac{\rho'\Omega}{2\pi}\iiint\frac{(c-x)^{n+1}\cos\phi d\rho d\cdot d\phi}{\{(z'-z)^2+\rho'^2-2(c-x)\rho'\cos\phi+(c-x)^2\}^{\frac{1}{2}}}$$

where the integral is to be taken throughout the volume of the filament

$$=\frac{\Omega\rho'}{\pi}\iint e^{-x\frac{d}{dc}-z\frac{d}{dz'}}\,dx\,dz\int_0^\pi\frac{c^{n+1}\cos\phi d\phi}{\{z'^2+\rho'^2-2c\rho'\cos\phi+c^2\}^{\frac{1}{2}}}$$

where the first integral is to be taken throughout any cross-section of the ring

$$=\frac{\Omega\rho'}{\pi}\iint e^{-r\nabla\cos(\theta-a)}\,rdrd\theta(c^n J) \qquad \ldots \quad (37)$$

where the integral is to be taken over any cross-section of the ring.

It will be proved that even in this case. the cross-section is not circular but is given by the equation (4) if σ^4 and higher powers of σ be neglected, A_2 and A_3 being quantities of the order σ^2 and σ^3 respectively.[1]

$\therefore$ From (6) and (37).

$$\psi=\Omega\rho' a^2\Bigg[1+\frac{a^2\nabla^2}{8}+\frac{a^4\nabla^4}{192}+\frac{a^6\nabla^6}{9216}+\text{etc.}$$

$$+\frac{A_2}{4}a^2\nabla^2\cos 2a\left(1+\frac{a^2\nabla^2}{12}+\frac{a^4\nabla^4}{384}+\text{etc.}\right)$$

$$-\frac{A_3}{24}a^3\nabla^3\cos 3a\left(1+\frac{a^2\nabla^2}{16}+\text{etc.}\right)\Bigg]c^n J \qquad \ldots \quad (38)$$

Now, from (8)

$$\nabla^2 J=\left(\frac{1}{c}\frac{d}{dc}\right)J$$

$$\nabla^2 c^n J=c^n\nabla^2 J+2nc^{n-1}\frac{dJ}{dc}+n(n-1)c^{n-2}J$$

$$=c^n\left[(2n+1)\left(\frac{1}{c}\frac{d}{dc}\right)+\frac{n(n-1)}{c^2}\right]J \qquad \ldots \quad (39)$$

[1] See results (48) and (49).

Similarly,

$$\nabla^4 c^n J = c^n \left[(2n+1)(2n-1)\left(\frac{1}{c}\frac{d}{dc}\right)^2 + 2(2n-1)n(n-1)\frac{1}{c^2}\left(\frac{1}{c}\frac{d}{dc}\right) + \frac{n(n-1)(n-2)(n-3)}{c^4}\right] J \quad \ldots \quad (39A)$$

$$\nabla^6 c^n J = c^n \left[(2n+1)(2n-1)(2n-3)\left(\frac{1}{c}\frac{d}{dc}\right)^3 + \frac{3(2n-1)(2n-3)n(n-1)}{c^6}\left(\frac{1}{c}\frac{d}{dc}\right)^2 + 3(2n-3)n(n-1)(n-2)(n-3)\frac{1}{c^4}\left(\frac{1}{c}\frac{d}{dc}\right) + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{c^6}\right] J \quad \ldots \quad (39B)$$

$$\nabla^8 c^n J = c^n \left[(2n+1)(2n-1)(2n-3)(2n-5)\left(\frac{1}{c}\frac{1}{dc}\right)^4 + \frac{4(2n-1)(2n-3)(2n-5)n(n-1)}{c^2}\left(\frac{1}{c}\frac{d}{dc}\right)^3 + \ldots + \frac{n(n-1)(n-2)\ldots(n-7)}{c^8}\right] J \quad \ldots \quad (39C)$$

etc. etc.

$$a^2\nabla^2 \cos 2\alpha\left(1+\frac{a^2\nabla^2}{12} + \text{etc.}\right)(Jc^n)$$

$$= a^2\left(2\frac{d^2}{dc^2} - \nabla^2\right)\left(1+\frac{a^2\nabla^2}{12} + \text{etc.}\right)(Jc^n)$$

$$= c^n\left[(2n+1)a^2\left(\frac{1}{c}\frac{d}{dc}\right) + 2a^2c^2\left(\frac{1}{c}\frac{d}{dc}\right)^2 + \frac{a^4c^2}{6}(2n+1)\left(\frac{1}{c}\frac{d}{dc}\right)^3\right] J \quad \ldots \quad (39D)$$

neglecting σ^2 and higher powers of σ.

Also $$a^3\nabla^3\cos 3\alpha\left(1+\frac{a^2\nabla^2}{16}+\text{etc.}\right)(c^n\mathrm{J})$$

$$=a^3\left(4\frac{d^3}{dc^3}-3\frac{d}{dc}\nabla^2\right)\left(1+\frac{a^2\nabla^2}{16}+\text{etc.}\right)(\mathrm{J}c^n)$$

$$=4c^na^3c^3\left(\frac{1}{c}\frac{d}{dc}\right)^3\mathrm{J} \qquad .. \quad (39\mathrm{E})$$

neglecting σ and higher-powers of σ.

Also $k=\iint 2\omega r dr d\theta$ taken over any cross-section

$$=2\Omega c^n a^2\pi\left\{1+\frac{n(n-1)}{8}\sigma^2\right\} \qquad \ldots \quad (40)$$

neglecting σ^4 and higher powers of σ.

Hence, always neglecting s^4 and higher powers of s, we have from (38) after a little simplification with the help of above equations, that ψ at any point (r,θ) outside the filament and very near to it,

$$=\Omega\rho' a^2c^n\Bigg[\left\{1+\frac{n(n-1)}{8}\sigma^2\right\}\mathrm{J}+\left\{\frac{2n+1}{8}a^2+\frac{a^4(2n-1)n(n-1)}{96c^2}\right.$$

$$\left.+\frac{2n+1}{4}\mathrm{A}_2a^2\right\}\frac{1}{c}\frac{d\mathrm{J}}{dc}+\left\{\frac{a^4}{192}(2n+1)(2n-1)\right.$$

$$\left.+\frac{\mathrm{A}_2a^2c^2}{2}\right\}\left(\frac{1}{c}\frac{d}{dc}\right)^2\mathrm{J}+\left\{\frac{a^6}{9216}(2n+1)(2n-1)(2n-3)\right.$$

$$\left.-\frac{\mathrm{A}_3}{6}a^3c^3+\mathrm{A}_2\frac{a^4c^2}{24}(2n+1)\right\}\left(\frac{1}{c}\frac{d}{dc}\right)^3\mathrm{J}\Bigg] \qquad \ldots \quad (41)$$

Hence, substituting the values of $\frac{\rho'}{c}\mathrm{J}, \left(\frac{1}{c}\frac{d}{dc}\right)\mathrm{J}.\frac{\rho'}{c}$ etc. from (15) to (19) in (41) ψ at any point on the surface of the ring

$$=\frac{kc}{2\pi}\Bigg[\left\{\text{const}+\left(-\frac{\lambda+1}{2}+\frac{3\lambda+5}{64}\sigma^2-\frac{\lambda}{4}\mathrm{A}_2\right)\sigma\cos\theta\right.$$

$$\left.-\left(\frac{\lambda\sigma^2}{16}+\mathrm{A}_2\right)\cos 2\theta-\left(\frac{3\lambda-1}{192}\sigma^3+\mathrm{A}_3+\frac{\lambda\sigma\mathrm{A}_2}{4}\right)\cos 3\theta\right\}$$

$$+\frac{\sigma}{8}\left\{(2n+1)-\frac{(2n+5)n(n-1)}{24}\sigma^2+2(2n+1)\mathrm{A}_2\right\}$$

$$\times\left\{\text{const}+\left(-1+\frac{4\lambda+1}{32}\sigma^2+\frac{A_2}{2}\right)\cos\theta+\frac{\sigma}{4}\cos2\theta\right.$$

$$\left.+\left(\frac{A_2}{2}+\frac{\sigma^2}{32}\right)\cos3\theta\right\}+\left\{\frac{\sigma^2}{192}(4n^2-1)+\frac{A_2}{2}\right\}$$

$$\times\left\{\text{const}-\frac{\sigma}{4}\cos\theta+\left(1-\frac{\sigma^2}{4}\right)\cos2\theta-\frac{\sigma}{4}\cos3\theta\right\}$$

$$+\left\{\frac{\sigma^3}{4608}(4n^2-1)(2n-3)-\frac{A_3}{3}+\frac{A_2\sigma(2n+1)}{12}\right\}(-\cos3\theta)\Big]$$

$$\doteq\frac{kc}{2\pi}\Big[\text{const}-\left\{\frac{4\lambda+2n+5}{8}+A_2\frac{4\lambda+6n+5}{16}\right.$$

$$\left.-\frac{12(n+2)\lambda+4n^3+4n^2-7n+32}{384}\sigma^2\right\}\sigma\cos\theta-\left\{\frac{A_2}{2}\right.$$

$$\left.+\frac{12\lambda-5-12n-4n^2}{192}\sigma^2\right\}\cos2\theta-\left\{\frac{2A_3}{3}+\frac{12\lambda+2n+7}{48}A_2\sigma\right.$$

$$\left.+\frac{72\lambda+8n^3+12n^2-38n-45}{4608}\sigma^3\right\}\cos3\theta\Big]. \quad \ldots \quad (42)$$

Since, the ring moves with a velocity of translation V parallel to z-axis,

$\psi=\frac{V\rho^2}{2}$ +const. on the surface of the ring

$$=\frac{Vc^2}{2}\left[\text{const}-(2+A_2)\sigma\cos\theta+\frac{\sigma^2}{2}\cos2\theta-\sigma A_2\cos3\theta\right] \text{ from (4).} \quad (43)$$

Since, the equation (43) is to hold good for all values of θ, we have from (42) and (43) by equating the co-efficients of $\cos\theta$, $\cos2\theta$, $\cos3\theta$ respectively,

$$Vc\left(1+\frac{A_2}{2}\right)=\frac{k}{2\pi}\left\{\frac{4\lambda+2n+5}{8}+A_2\frac{4\lambda+6n+5}{16}\right.$$

$$\left.-\frac{12(n+2)\lambda+4n^3+4n^2-7n+32}{384}\sigma^2\right\} \quad \ldots \quad (44)$$

$$Vc\sigma^2=-\frac{k}{\pi}\left\{A_2+\frac{12\lambda-5-12n-4n^2}{96}\sigma^2\right\} \quad \ldots \quad (45)$$

$$Vc\sigma A_2=\frac{k}{\pi}\left\{\frac{2}{3}A_3+\frac{12\lambda+2n+7}{48}A_2\sigma\right.$$

$$\left.+\frac{72\lambda+8n^3+12n^2-38n-45}{4608}\sigma^3\right\} \quad \ldots \quad (46)$$

From (45) and (46) it is evident that A_2 and A_3 are of the order σ^2 and σ^3 respectively.

Also from (44) to a first approximation,

$$Vc=\frac{k}{16\pi}(4\lambda+2n+5) \qquad \ldots \quad (47)$$

Substituting this value of V in (45) and simplifying

$$A_2=-\frac{36\lambda+25-4n^2}{96}\sigma^2. \qquad \ldots \quad (48)$$

Again, substituting these values of V and A_2 given by (47) and (48) respectively, in (46), we have

$$A_3=-\frac{72\lambda(2n+5)+155+62n-20-8n^3}{3072}\sigma^3 \qquad \ldots \quad (49)$$

Also, dividing both sides of (44) by $\left(1+\frac{A_2}{2}\right)$ and then substituting the value of A_2 as given in (48), we have,

$$V=\frac{k}{\pi c}\left\{\frac{4\lambda+2n+5}{16}-\frac{12\lambda(2n+1)+2n^2+9n+16}{384}\sigma^2\right\} \qquad \ldots \quad (50)$$

From (4), (48) and (49), the (r, θ) equation of the cross-section is

$$r=a\left[1-\frac{36\lambda+25-4n^2}{96}\sigma^2\cos2\theta\right.$$
$$\left.-\frac{72\lambda(2n+5)+155+62n-20n^2-8n^3}{3072}\sigma^3\cos3\theta\right] \qquad \ldots \quad (51)$$

From (48), A_2 vanishes when

$$n=\pm\sqrt{\frac{36\lambda+25}{4}}.$$

Hence, it is obvious that the section of the ring may generally be regarded as circular if σ^2 and higher powers of σ are negligible. If, however,

$$n=\pm\sqrt{\frac{36\lambda+25}{4}},$$

the cross-section is circular correct to σ^2.

9. In Art 8, we have found out the expression for the velocity and the shape correct to σ^3 but the same process may be used to find them out correct to any higher powers of σ.

Verification of previous results.

10. Some particular cases are easily deducible from our general results. Thus,

(1) when $n=0$, from (50) and (51),

$$V=\frac{k}{\pi c}\left\{\frac{4\lambda+5}{16}-\frac{3\lambda+4}{96}\sigma^2\right\}$$

and $$r=a\left[1-\frac{36\lambda+25}{96}\sigma^2\cos 2\theta-\frac{360\lambda+155}{3072}\sigma^3\cos 3\theta\right].$$

These results are identical[1] with those already obtained by me.

(2) when $n=1$, from (50),

$$V=\frac{k}{\pi c}\left\{\frac{4\lambda+7}{16}-\frac{12\lambda+9}{128}\sigma^2\right\}$$

and from (51),

$$r=a\left[1-\frac{12\lambda+7}{32}\sigma^2\cos 2\theta-\frac{168\lambda+63}{1024}\sigma^3\cos 3\theta\right]$$

These results have already been obtained by Hicks[2] and Dyson.[3]

11. Now, let us deduce the results when the vorticity is given by (35) *viz.*, $\omega=\Omega\rho^{-2}$.

Putting $n=2$, from (50),

$$V=\frac{k}{\pi c}\left\{\frac{4\lambda+1}{16}+\frac{6\lambda-1}{64}\sigma^2\right\}$$

and from (51),

$$r=a\left[1-\frac{12\lambda+3}{32}\sigma^2\cos 2\theta-\frac{24\lambda+5}{1024}\sigma^3\cos 3\theta\right]$$

Hence, the cross-section of the ring gets elongated in the direction of motion also in this case.

Velocity of translation.

12. From (50), neglecting σ^2 and higher powers, we have

$$V=\frac{k}{16\pi c}(4\lambda+2n+5)$$

[1] See results (27) and (30).

[2] Loc. cit. Part I and II.

[3] Loc. cit. Part II.

Now, (1) when $n=0$, evidently

$$V=\frac{k}{4\pi c}\left(\log\frac{8c}{a}-\frac{3}{4}\right)$$

(2) when $n=1$,

$$V=\frac{k}{4\pi c}\left(\log\frac{8c}{a}-\frac{1}{4}\right)$$

(3) when $n=-2$,

$$V=\frac{k}{4\pi c}\left(\log\frac{8c}{a}-\frac{7}{4}\right)$$

(4) To find n such that

$$V=\frac{k}{4\pi c}\left(\log\frac{8c}{a}-1\right)=\frac{k}{4\pi c}(\lambda+1),$$

we have

$$\lambda+1=\frac{4\lambda+2n+5}{4} \quad i.e., \; n=-\frac{1}{2} \qquad \dots \quad (52)$$

Hence, it is when vorticity $\propto \rho^{-\frac{1}{2}}$ that the velocity is given by

$$\frac{k}{4\pi c}\left(\log\frac{8c}{a}-1\right).$$

The expression for the velocity and shape of the cross-section in this case, correct to σ^3, may be easily deduced from (50) and (51).

Fluted Oscillations of the vortex ring
Vorticity $\propto \rho^n$

13. Let the central circle of the ring have moved a distance z_0 from the plane of $x\,y$ and let the cross-section in the disturbed position be given by

$$r=a\{1+\sum(\alpha_m\sin m\theta+\beta_m\cos m\theta)\} \qquad \dots \quad (53)$$

Then, proceeding exactly as in Art 3 and 8, it may be obtained without much difficulty that for a ring of cross-section given by (53)

$$\psi=\frac{k\rho'}{2\pi}\left[J+\frac{(2n+1)a^2}{8}\ \frac{1}{c}\ \frac{dJ}{dc}+\text{etc.}\right.$$

$$\left.+\Sigma\ \frac{a^m}{mr^m}\ (\alpha_m\sin m\theta+\beta_m\cos m\theta)\right]$$

$$=\frac{kc}{2\pi}\left[l-\frac{l+1}{2}\ \frac{r}{c}\cos\theta-\frac{2n+1}{8}\ \frac{a^2}{cr}\cos\theta\right.$$

$$\left.+\Sigma\ \frac{a^m}{mr^m}(\alpha_m\sin m\theta+\beta_m\cos m\theta)\right] \quad \dots \quad (54)$$

Now,

$$\frac{dr}{dt}=\frac{\partial r}{\partial t}+\frac{\partial r}{\partial z_0}\ \frac{\partial z_0}{\partial t}+\frac{\partial r}{\partial c}\ \frac{\partial c}{\partial t}$$

∴ From (53) on the surface of the ring

$$\frac{dr}{dt}=\{1+\Sigma(\alpha_m\sin m\theta+\beta_m\cos m\theta)\}\dot{a}+a\Sigma\{m\dot{\theta}(\alpha_m\cos m\theta$$

$$-\beta_m\sin m\theta)+(\dot{\alpha}_m\sin m\theta+\dot{\beta}_m\cos m\theta)\} \quad \dots \quad (55)$$

But

$$\frac{\partial r}{\partial t}=\frac{1}{\rho r}\ \frac{\partial\psi}{\partial\theta}=\frac{k}{2\pi(c-r\cos\theta)}\left\{\frac{l+1}{2}\sin\theta+\frac{2n+1}{8}\ \frac{a^2}{r^2}\sin\theta\right.$$

$$\left.+\Sigma\ \frac{a^m}{r^{m+1}}\ (\alpha_m\cos m\theta-\beta_m\sin m\theta)\right\}$$

$$=\frac{k}{2\pi c}\left\{\frac{4\lambda+2n+5}{8}\ \sin\theta\right.$$

$$\left.+\Sigma\frac{c}{a}\ (\alpha_m\cos m\theta-\beta_m\sin m\theta)\right\}$$

nearly when $r=a$

$$r\dot{\theta}=-\frac{1}{\rho}\frac{\partial\psi}{\partial r}=\frac{kc}{2\pi(c-r\cos\theta)}\frac{1}{r}\text{ approximately.}$$

$$\therefore\ \dot{\theta}=\frac{k}{2\pi a^2}\quad\text{nearly when } r=a$$

Also it is easy to see that

$$\frac{\partial r}{\partial c}=\cos\theta\,;\quad \frac{\partial r}{\partial z_0}=-\sin\theta,\quad \frac{\partial\theta}{\partial c}=-\frac{\sin\theta}{r},\quad \frac{\partial\theta}{\partial z_0}=-\frac{\cos\theta}{r}$$

Hence from (55),

$$\frac{k}{2\pi c}\left\{\frac{4\lambda+2n+5}{8}\sin\theta+\Sigma\frac{c}{a}(\alpha_m\cos m\theta-\beta_m\sin m\theta\right\}$$

$$-\sin\theta\frac{dz_0}{dt}+\cos\theta\dot{c}=\dot{a}+\Sigma\left\{\sin m\theta\left(\dot{a}\alpha_m+a\dot{\alpha}_m-\frac{km}{2\pi a}\beta_m\right)\right.$$

$$\left.+\cos m\theta\left(\dot{a}\beta_m+a\dot{\beta}_m+\frac{km}{2\pi a}\alpha_m\right)\right\}.$$

This equation gives

$$\dot{a}=0,\quad \dot{c}=0$$

$$\dot{z}_0=\mathrm{V}=\frac{k}{4\pi c}\left(\lambda+\frac{2n+5}{4}\right)=\frac{k}{4\pi c}\left(\log\frac{8c}{a}+\frac{2n-3}{4}\right)$$

$$\dot{\alpha}_m=\frac{k\beta_m}{2\pi a^2}(m-1)$$

$$\dot{\beta}_m=-\frac{k\alpha_m}{2\pi a^2}(m-1)$$

i.e.,

$$\ddot{\alpha}_m+\frac{k^2(m-1)^2}{4\pi^2a^4}\alpha_m=0$$

$$\ddot{\beta}_m+\frac{k^2(m-1)^2}{4\pi^2a^4}\beta_m=0$$

$\therefore$ The oscillation is simple harmonic, the period being

$$\frac{4\pi^2 a^2}{k(m-1)}.$$

14. Conclusion.—In the preceding articles, I have studied the motion of a single vortex ring in an incompressible fluid. In a similar way the motion of any number of vortex rings can be investigated when the vorticity $\propto \rho^n$ and the results obtained by Dyson[1] may be easily deduced therefrom by putting $n=1$.

The motion of vortex rings of finite section in a compressible fluid as well as the distribution of vorticity for which the cross-section of rings is exactly circular will be discussed in a subsequent paper.

[1] Loc. cit. ante.

The Steering of an Aeroplane in a Horizontal Circle.

BY

Nalinikanta Basu.

Let us start by writing down the general equations of motion of Rigid Dynamics. Taking the Centre of Mass of the aeroplane as the origin of co-ordinates and 3 rectangular axes fixed relatively to the aeroplane and moving with it in space and using the following notations :—

W, weight of the aeroplane,

A, B, C, moments of inertia about the axes,

D, E, F, corresponding products of inertia,

u, v, w, components of translational vel,

p, q, r, ,, of angular vel,

h_1, h_2, h_3, ,, of angular momentum,

we have the following equations of motion :—

$W\left(\frac{du}{gdt}+\frac{qw}{g}-\frac{rv}{g}\right)$= Acc. force along the x-axis and 2 similar equations,

also, $\frac{dh_1}{gdt}+\frac{qh_3}{g}-\frac{rh_2}{g}$ = Acc. torque about the x-axis and 2 similar equations and

$$h_1=Ap-Fq-Er$$

$$h_2=Bq-Dr-Fp$$

$$h_3=Cr-Ep-Dq.$$

In the first place, let the aeroplane be flying steadily in a horizontal straight line. Let this be the axis of x (the line parallel to the line of flight and passing through the C G) and a line drawn vertically downwards through the C G, the y-axis and a horizontal line perpendicular to these the z-axis.

If the aeroplane be turned in any other directions the following angular co-ordinates will specify them.

Starting from an initial position, let us rotate the aeroplane about y-axis through an angle ψ and then about the new position of the z-axis through an angle θ and lastly about the final position of the x-axis through ϕ. The cosines of the angles between the old axes $x_0\ y_0\ z_0$ and the new axes $x_1\ y_1\ z_1$ are given by

$$\begin{array}{cccc} & x_1 & y_1 & z_1 \\ x_0 & \cos\theta\cos\psi, & \sin\phi\sin\psi-\cos\phi\cos\psi\sin\theta, & \cos\phi\sin\psi+\sin\phi\cos\psi\sin\theta \\ y_0 & \sin\theta, & \cos\theta\cos\phi, & -\cos\theta\sin\phi \\ z_0 & -\cos\theta\sin\psi, & \sin\phi\cos\psi+\cos\phi\sin\psi\sin\theta, & \cos\phi\cos\psi-\sin\phi\sin\psi\sin\theta \end{array}$$

and the angular velocities p, q, r are given in terms of $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$

$$p=\dot{\phi}+\dot{\psi}\sin\theta$$

$$q=\dot{\theta}\sin\phi+\dot{\psi}\cos\theta\cos\phi$$

$$r=\dot{\theta}\cos\phi-\dot{\psi}\cos\theta\sin\phi.$$

The impressed forces and couples are due to (*i*) gravity (*ii*) the propeller thrust (*iii*) air resistances.

The components of gravity along the axes are

$$W\sin\theta,\ W\cos\theta\cos\phi,\ -W\cos\theta\sin\phi,$$

the corresponding moments all vanishing.

The propeller thrust is assumed to act along a line parallel to the x-axis and at a point on the y-axis distant h, from the origin, then the components of thrust are

Point of application, 0, h, 0

Force, H, 0, 0

Torque, 0, 0, $-Hh$.

For the components of air resistances we assume that they reduce to X, Y, Z and L, M, N and these are taken positive when they tend to retard the corresponding motions of translations and rotations.

Hence the equations of motion are in the case of a symmetrical aeroplane (in which D=E=0),

$$W\left(\frac{du}{gdt}+\frac{qw}{g}-\frac{rv}{g}\right)=W\sin\theta+H-X$$

$$W\left(\frac{dv}{gdt}+\frac{ru}{g}-\frac{pw}{g}\right)=W\cos\theta\cos\phi-Y$$

$$W\left(\frac{dw}{gdt}+\frac{pv}{g}-\frac{qu}{g}\right)=-W\cos\theta\sin\phi-Z$$

$$A\frac{dp}{gdt}-F\frac{dq}{gdt}+(C-B)\frac{rq}{g}+F\frac{pr}{g}=-L$$

$$B\frac{dq}{gdt}-F\frac{dp}{gdt}+(A-C)\frac{pr}{g}-F\frac{qr}{g}=-M$$

$$C\frac{dr}{gdt}+(B-A)\frac{pr}{g}-F\frac{p^2-q^2}{g}=-HL-N.$$

As the aeroplane is supposed to be steering steadily in a horizontal circle then initially $u=U-aQ$, $q=Q$ where Q is the angular velocity of the aeroplane in the circle and 'a' its redius and $v=w=p=r=\theta=\phi=0$, then the initial conditions of equilibrium are

$$0=H-X_0$$

$$0=W-Y_0$$

$$-\frac{W}{g}aQ^2=-Z_0$$

$$0=-L_0$$

$$0=-M_0$$

$$F\frac{Q^2}{g}=-Hh-N_0.$$

where X_0 Y_0 Z_0 L_0 M_0 N_0 are the initial values of the resistances.

Suppose then the angular velocity of the aeroplane is suddenly given an increment so that it becomes $Q+q$, the path of the aeroplane being still horizontal. So 'v' the vertical velocity is still zero. Let us also suppose that w, p, q, r, θ, ϕ be corresponding small increments, then

$U+u=(a+\rho)(Q+q)=U+aq+\rho Q$, where $a+\rho$ is the new radius

$w=\dot{\rho}$, $\dot{\theta}=r$, $\dot{\phi}=p$.

In the first instance we assume $F=0$, the equations are

$$W\left(\frac{du}{gdt}+\frac{Qw}{g}\right)=W\theta+H-X_0-uX_u-rX_r$$

$$Wr\frac{U}{g}=W-Y_0-uY_u-rY_r$$

$$W\left(\frac{dw}{gdt}-q\frac{U}{g}\right)=-W\phi-Z_0-wZ_w-pZ_p-qZ_q$$

$$A\frac{dp}{gdt}+(C-B)\frac{rq}{g}=-L_0-wL_w-pL_p-qL_q$$

$$B\frac{dq}{gdt}+(A-C)\frac{pr}{g}=-M_0-wM_w-pM_p-qM_q$$

$$C\frac{dr}{gdt}+(B-A)\frac{pq}{g}=-Hh-N_0-uN_u-rN_r.$$

Substituting in these $u=aq+\rho Q$, $w=\dot{\rho}$, $r=\dot{\theta}$, $p=\dot{\phi}$

$$W\left(\frac{a}{g}\frac{dq}{dt}+\frac{Q}{g}\frac{d\rho}{dt}\right)+\frac{W}{g}Q\frac{d\rho}{dt}=W\theta-(aq+\rho Q)X_u-\frac{d\theta}{dt}.X_r$$

$$\frac{W}{g}aQ.\frac{d\theta}{dt}=-(aq+Q\rho)Y_u-\frac{d\theta}{dt}.Y_r$$

$$A\frac{dp}{gdt}+(C-B)\frac{Q}{g}.\frac{d\theta}{dt}=-\frac{d\rho}{dt}L_w-pL_p-qL_q$$

$$B\frac{dq}{gdt}=-\frac{d\rho}{dt}M_w-pM_p-qM_q.$$

To investigate these oscillations we assume ρ, q, θ, p each proportional to $e^{\lambda t}$ and substitute these values in the above equations and arranging we get

$$\left(\frac{W}{g}a\lambda+aX_u\right)q+\left(2\frac{W}{g}Q\lambda+QX_u\right)\rho+(\lambda X_r-W)\theta+0.p=0$$

$$aY_u.q+QY_u.\rho+\lambda\left(Y_r+\frac{W}{g}aQ\right)\theta+0.p=0$$

$$\left(\frac{B}{g}\lambda+M_q\right)q+\lambda M_w\rho+0.\theta+M_p.p=0$$

$$L_q.q+\lambda.L_w.\rho+(C-B)\frac{Q}{g}\lambda.\theta+\left(\frac{A}{g}\lambda+L_p\right)p=0.$$

Eliminating q, ρ, θ, p we get the determinant

$$\begin{vmatrix} \frac{W}{g}\, a\,\lambda + aX_u, & 2\,\frac{W}{g}\, Q\lambda + QX_u, & \lambda X_r - W, & 0 \\ a\,Y_u, & Q.Y_u, & (Y_r + \frac{W}{g}\, aQ)\,\lambda, & 0 \\ L_q, & \lambda.L_w, & (C-B)\,\frac{Q}{g}\,\lambda, & \frac{A}{g}\,\lambda + L_p \\ \frac{B}{g}\lambda + M_q, & \lambda\, M_w, & 0. & M. \end{vmatrix} = 0.$$

Developing the determinant in powers of λ we get an equation of the 4th degree in λ

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0$$

where $$A = (Y_r + \frac{W}{g}\, a\, Q)\left(a\, M_w - 2\,\frac{BQ}{g}\right)\frac{AW}{g^2}$$

$$B = (Y_r + \frac{W}{g}\, aQ)\,(M_w\, L_p - M_p\, L_w)\,\frac{W}{g}\, a + \frac{Aa}{g}\, M_w(X_u Y_r - X_r Y_u)$$

$$- \frac{AB}{g^2}\, Q\,(X_u\, Y_r - X_r\, Y_u) + \frac{W}{g}\, aQ.\frac{A}{g}\, X_u\left(a\, M_w - \frac{BQ}{g}\right)$$

$$-\left(Y_r + \frac{W}{g}\, aQ\right)(AM_q + BL_p)\,\frac{2W}{g^2}\, Q$$

$$C = -(C-B)\,\frac{Q}{g}\, M_p\, Y_u\,\frac{AW}{g}\, Q + (Y_r + \frac{W}{g}\, aQ)\,(M_w L_p - M_p\, L_w)\, aX_u$$

$$+ \frac{2WQ}{g}\,(Y_r + \frac{W}{g}\, aQ)\,(L_q\, M_p - L_q\, M_q) - \frac{Q}{g}\,(AM_q + BL_p)\,(X_u Y_r$$
$$- X_r\, Y_u)$$

$$+ a\, Y_u\, X_r\,(L_w\, M_p - L_p\, M_w) - (AM_q + BL_p)\,\frac{W}{g}\, aQ^2\, X_u$$

$$+ \frac{A}{g}\, aM_w.WY_u + \frac{ABW}{g^2}\, QY_u$$

$$D = -QX_u\,(Y_r + \frac{W}{g}\, aQ)\,(L_p M_q - L_q M_p) - WaY_u\,(L_w M_p - L_p M_w)$$

$$- Q.X_r Y_u\,(L_q\, M_p - L_p\, M_q) - \frac{W}{g}\, QY_u\,(AM_q + BL_p)$$

$$E = WQY_u\,(L_q\, M_p - M_q\, L_p).$$

The condition of stability require that all the four roots of the biquadratic equation for λ shall have their real part negative. This follows from the assumption that the disturbances ρ, q, θ, p are all proportional to $e^{\lambda t}$ in a typical oscillation. If λ_1 λ_2 λ_3 λ_4 are the roots the expressions for ρ, q, θ, p take the form

$$a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + a_3 e^{\lambda_3 t} + a_4 e^{\lambda_4 t}$$

a_1 a_2 a_3 a_4 being constant co-efficients determined by the initial conditions.

The conditions that the roots of a biquadratic equation shall all have their real part negative and thus indicate stability of steady motion is given by Routh. The condition is

A, B, C, D, E and F where $F=BCD-AD^2-B^2E$, shall all have the same sign.

Now let us examine the behaviour in the above particular case of the aeroplane which Bryan has shown to possess inherent stability of great range. This is the system with 2 raised fins at the same height—two fins of areas T_1 and T_2 (of total area T) one in front and other in the rear of the C G of the system and both above the x-axis in $x-y$ plane with the y of the C P equal and their joint C P in a line through the C. G. of the system perpendicular to the main plains.

Let x, y be the co-ordinates of the Centre of mean Position (or centre of Pressure) of the 2 fins, and M_1 M_2 and P the moments and the products of inertia of the axes of the fins with respect to axes parallel to the co-ordinate axes through (x, y) we get from Bryan since $M_1=P=0$, in this case for the fins

$$Z_w=KTU,\ Z_p=KTUy,\ Z_q=-KTUx$$

$$L_w=KTUy,\ L_p=KTUy^2,\ L_q=-KTUxy$$

$$M_w=-KTUx,\ M_p=-KTUxy,\ M_q=KU\ (Tx^2+M_2)$$

and by Lanchesters' Fin Resolution, $M_2=\dfrac{T_1\ T_2}{T_1+T_2}$ (distance bet. fins)^2.

Let us assume a small so that $x=0$, and also $F=0$, *i.e.*, the x—axis is a principle axis. Then

$$Z_w=KTU,\ Z_p=KTUy,\ Z_q=0$$

$$L_w=KTUy,\ L_p=KTUy^2,\ L_q=0$$

$$M_w=0\ M_p=0,\ M_q=kUM_2.$$

The above are the resistance derivatives due to the 2 fins only, the resistance derivatives for a main plane at an angle a and a rudder plane,

$$Z_w=0,\ Z_p=0,\ Z_q=0$$

$$L_w=0,\ L_p=\mathrm{kUI}\cos^2 a,\ L_q=-2k\mathrm{UI}\sin a\cos a$$

$$M_w=0,\ M_p=k\mathrm{UI}\sin a\cos a,\ M_q=2k\mathrm{UI}\sin^2 a.$$

Hence the whole resistance derivatives are

$$X_u=2ks_1\ \mathrm{U}\sin^2 a,\ X_v=ks_1\ \mathrm{U}\sin a\cos a,\ X_r=0$$

$$Y_u=2ks_1\ \mathrm{U}\sin a\cos a,\ Y_v=ks_1\ \mathrm{U}\cos^2 a+ks_2\ \mathrm{U},\ Y_r=-ks_2\ \mathrm{U}l$$

$$N_u=0,\ Nv=-ks_2\ \mathrm{U}l,\ N_r=ks_2\mathrm{U}l^2$$

$$Z_w=\mathrm{KTU},\ Z_p=k\mathrm{UT}y,\ Z_q=0$$

$$L_w=k\mathrm{UT}y,\ L_p=k\mathrm{UT}y^2,\ L_q=-2k\mathrm{UI}\tan a$$

$$M_w=0,\ M_p=-k\mathrm{UI}\sin a\cos a,\ M_q=k\mathrm{UM}_2.$$

Substituting these values in A, B, C, D and E we get

$$\frac{\mathrm{A}}{a^2\,\mathrm{Q}^2}=\frac{2\mathrm{ABW}}{ag}\left(ks_2 l-\frac{\mathrm{W}}{g}\right)\frac{\mathrm{AW}}{g^2}$$

$$\frac{\mathrm{B}}{a^3\,\mathrm{Q}^3}=\left(ks_2 l-\frac{\mathrm{W}}{g}\right)k^2\mathrm{I}\ \frac{\mathrm{W}}{g}\ a\sin a\cos a\,\mathrm{T}y$$

$$+\frac{2k^2\mathrm{AB}}{ag^2}\ s_1\sin^2 a\,(s_1\cos^2 a+s_2)$$

$$-\frac{\mathrm{WAB}}{ag^3}\ 2ks_1\sin^2 a-\frac{2\mathrm{W}}{ag^2}\ k\left(ks_2 l-\frac{\mathrm{W}}{g}\right)(\mathrm{AM}_2+\mathrm{BT}y^2)$$

$$\frac{\mathrm{C}}{a^4\,\mathrm{Q}^4}=2\left(ks_2 l-\frac{\mathrm{W}}{g}\right)k^3s_1\mathrm{I}\sin^3 a\cos a\,\mathrm{T}y\ a$$

$$-(\mathrm{C}-\mathrm{B})\ \frac{4\mathrm{WI}}{ag^2}\ k^2s_1\sin^2 a\cos^2 a$$

$$+\left(ks_2l-\frac{W}{g}\right)(M_2\, Ty^2+2I^2\sin^2 a)\frac{2W}{ag}k^2$$

$$-\frac{W}{ag}2k^2s_1\,(AM_2+BTy^2)\sin^2 a+\frac{2k^3}{ag}(AM_2+BTy^2)\,s_1s_2l\sin^2 a$$

$$-\frac{AB}{a^3gQ}ks_2lW$$

$$\frac{D}{a^5Q^5}=\frac{2k^2s_1}{a}\left(ks_2l-\frac{W}{g}\right)(QI^2\sin^2 a+M_2Ty^2)\sin^2 a$$

$$-\frac{W}{aQ^2}2k^3s_1\;ITy\sin^2 a\cos^2 a-\frac{W}{a^3Q^2}2k^2s_1\,(AM_2+BTy^2)\sin a\cos a$$

$$\frac{E}{a^6\,Q^6}=-\frac{2Wk^3}{a^3\,Q^2}(2I^2\sin^2 a+M_2Ty^2)\,s_1\sin a\cos a.$$

Now from the conditions of equilibrium $W=ks_1\,U^2\sin a\cos a$ $=ks_1\,a^2\,Q^2\sin a\cos a$ and substituting these values of W in the above quantities

$$\frac{A}{U^2}=-\frac{2ABW}{ag^3}\,(ks_1\,U^2\sin a\cos a/g-ks_2l)$$

$$\frac{B}{U^3}=-\frac{KW}{g}(ks_1\,U^2\sin a\cos a/g-ks_2l)\left\{KIaTy\sin a\cos a\right.$$

$$\left.-\frac{2(AM_2+BTy^2)}{ag}\right\}$$

$$-\frac{2kAB}{ag^2}s_1\sin^2 a\left\{\frac{W}{g}-k\,(s_1\cos^2 a+s_2)\right\}$$

$$\frac{C}{U^4}=-\left(\frac{ks_1U^2\sin a\cos a}{g}-ks_2l\right)(M_2Ty^2+2I^2\sin^2 a)\,\frac{2Wk^2}{ag}$$

$$-\frac{AB}{a^3gQ}ks_2lW-2\left(\frac{ks_1U^2\sin a\cos a}{g}-ks_2l\right)k^3s_1\mathrm{l}\sin^3 a$$

$$\times\cos a\cdot Tya-(C-B)\,\frac{4WI}{ag^2}k^2s_1\sin^2 a\cos^2 a$$

$$+\frac{2k^3}{ag}(AM_2+BTy^2)\,s_1s_2l\sin^2\alpha-\frac{W}{ag}\,2k^2s_1\,(AM_2+BTy^2)\sin^2\alpha$$

$$\frac{D}{U^5}=-\frac{2k^2s_1}{a}(ks_1U^2\sin\alpha\cos\alpha/g-ks_2l)\,(2I^2\sin^2\alpha+M_2\,Ty^2)\sin^2\alpha$$

$$-\frac{W}{aQ^2}\,2k^3s_1\,ITy\sin^2\alpha\cos^2\alpha-\frac{W}{a^3Q^2}\,2k^2s_1\,(AM_2+BTy^2)\sin\alpha\cos\alpha$$

$$\frac{E}{U^6}=-\frac{2k^4}{a^2}s_1\,(2I^2\sin^2\alpha+M_2Ty_2)\sin^2\alpha\cos^2\alpha.$$

Let 'l' be positive, *i.e.*, the rudder be behind the main planes and not in front. Then A, B, C, D and E will all have the same sign as A if

$$U^2>\frac{s_1}{s_2}\;\frac{lg}{\sin\alpha\cos\alpha}$$

and if α be a small quantity then

$$U^2>\frac{s_1}{s_2}\,lg\cot\alpha$$

and this value of u^2 satisfies the other condition of stability $F=BCD-EB^2-AD^2<0$.

Hence for stability of an aeroplane moving in a horizontal circle with its rudder plane behind its main planes,

$$U^2>\frac{s_1}{s_2}\,lg\cot\alpha.$$

If l be negative, *i.e.*, if the rudder plane be in front of the main planes and if α be small so that we may neglect $\sin^2\alpha$ and higher powers of $\sin\alpha$

$$A/U^2=-\frac{2ABW}{ag^2}\left(\frac{W}{g}+ks_2l\right)$$

$$B/U^3=-\frac{kW}{g}\left(\frac{W}{g}+ks_2l\right)\left\{kIaTy\sin\alpha\cos\alpha\right.$$

$$\left.-\frac{2}{ag}(AM_2+BTy^2)\right\}$$

$$C/U^4 = -\frac{2Wk^2}{ag}\left(\frac{W}{g} + ks_2 l\right) M_2 T y^2$$

$$+\frac{AB}{a^3 g^2} k^2 s_1 s_2 l \sin \alpha \cos \alpha$$

$$D = 0$$

$$E = 0 \text{ and } F = 0.$$

Thus two roots of the biquadratic vanish. This denotes instability.

The Radius of a Circle in Homogeneous Co-ordinates

BY

Mohitmohan Ghosh, M.Sc.

The object of the present paper is to obtain in elegant forms the radius of a circle represented by the general equation of the second degree in homogeneous co-ordinates and the conditions for a circle as well as the co-ordinates of its centre. For the sake of simplicity the areal system of co-ordinates has been used in all the following investigations. The methods adopted are all elementary and I have purposely omitted to make use of the principle of Invariants.

First Method:

The distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in areal co-ordinates is given by

$$r^2 = -a^2(y_1-y_2)(z_1-z_2) - b^2(z_1-z_2)(x_1-x_2) - c^2(x_1-x_2)(y_1-y_2)$$

$\therefore$ the equation of a circle with centre at (x_1, y_1, z_1) and radius r is

$$r^2 + a^2(y-y_1)(z-z_1) + b^2(z-z_1)(x-x_1) + c^2(x-x_1)(y-y_1) = 0$$

or written in the homogeneous form

$$a^2yz + b^2zx + c^2xy - (x+y+z)\{a^2(yz_1+y_1z) + b^2(zx_1+z_1x) + c^2(xy_1+x_1y)\}$$
$$+(x+y+z)^2(a^2y_1z_1 + b^2z_1x_1 + c^2x_1y_1 + r^2) = 0.$$

If this be the same as the equation

$$ux^2 + vy^2 + wz^2 + 2u'yz + 2v'zx + 2w'xy = 0$$

by comparing the co-efficients we get

$$\frac{P-(b^2z_1+c^2y_1)}{u} = \frac{P-(c^2x_1+a^2z_1)}{v} = \frac{P-(b^2x_1+a^2y_1)}{w}$$

(1) (2) (3)

$$\frac{2P-(b^2+c^2-a^2)x_1}{2u'} = \frac{2P-(c^2+a^2-b^2)y_1}{2v'} = \frac{2P-(a^2+b^2-c^2)z_1}{2w'}$$
$$=k' \text{ (say)}$$

(4) (5) (6)

Where P stands for $a^2y_1z_1+b^2z_1x_1+c^2x_1y_1+r^2$.

From (2), (3) and (4) we at once get

$$k'=-\frac{a^2(x_1+y_1+z_1)}{v+w-2u'}=-\frac{a^2}{v+w-2u'}$$

Hence from symmetry we at once have as the conditions for a circle

$$\frac{v+w-2u'}{a^2}=\frac{w+u-2v'}{b^2}=\frac{u+v-2w'}{c^2}=k.$$

To find the centre we have to solve the above equations for (x_1, y_1, z_1).

From (1) (2) and (3) we have each of the above ratios equal to

$$\frac{c^2a^2\left(\frac{z_1}{c^2}+\frac{z_1}{a^2}\right)-b^2c^2\left(\frac{y_1}{b^2}+\frac{z_1}{c^2}\right)}{u-v}$$

$$=\frac{a^2b^2\left(\frac{x_1}{a^2}+\frac{y_1}{b^2}\right)-c^2a^2\left(\frac{z_1}{c^2}+\frac{x_1}{a^2}\right)}{v-w}$$

Hence we have the following equation for the centre:

$$xb^2c^2\left\{\frac{w-u}{b^2}+\frac{u-v}{c^2}\right\}+yc^2a^2\left\{\frac{u-v}{c^2}+\frac{v-w}{a^2}\right\}$$

$$+a^2b^2\left\{\frac{v-w}{a^2}+\frac{w-u}{b^2}\right\}=0 \qquad \dots \quad \text{(A)}$$

From (4), (5) and (6) we have similarly

$$x\frac{\cos A}{a}(v'-w')+y\frac{\cos B}{b}(w'-u')+z\frac{\cos C}{c}(u'-v')=0 \qquad \dots \quad \text{(B)}$$

The centre may now be found by solving (A) and (B).

Geometric meanings of (*A*) and (*B*).

It is at once seen that the above two equations will always give the centre except in the case of the circum-circle and the polar circle. Hence it may be guessed that (A) is connected in some way with the circum-circle and (B) with the polar circle. It is also seen that (A) passes through the circum-centre and (B) through the ortho-centre or the centre of the polar circle. Therefore they must be the perpendiculars upon the radical axes of the given circle with the circum-circle and the polar circle from the circum-centre and the ortho-centre respectively. This may be verified directly as follows;

The general equation may be written in the form

$$(ux+vy+wz)(x+y+z)-\{(v+w-2u')\,yz+(w+u-2v')zx$$
$$+(u+v-2w')\,xy\}=0$$

i.e., $\dfrac{1}{k}(ux+vy+wz)(x+y+z)-(a^2yz+b^2zx+c^2xy)=0$

$\therefore$ the radical axis of the circum-circle and the given circle is

$$ux+vy+wz=0.$$

The perpendicular upon this from the circum-centre (a cosA, b cosB, c cosC) is

$$\begin{vmatrix} ua^2-vab\cos C-wca\cos B, & vb^2-wbc\cos A-uab\cos C, & wc^2-vbc\cos A-uca\cos B \\ a\cos A & b\cos B & c\cos C \\ x & y & z \end{vmatrix}=0$$

The co-efficient of x is $b\cos B\,(wc^2-vbc\cos A-uca\cos B)$

$$-c\cos C\,(vb^2-wbc\cos A-uab\cos C)$$

$=abcu\,(\cos^2C-\cos^2B)-b^2cv\,(\cos C+\cos A\cos B)$

$$+bc^2w\,(\cos B+\cos C\cos A)$$

$=abcu\,(\sin^2B-\sin^2C)-b^2cv\sin A\sin B+bc^2w\sin C\sin A$

$=\dfrac{abc}{4R^2}\{u\,(b^2-c^2)-vb^2+wc^2\}$ where R=the circum-radius

$=\dfrac{abc}{4R^2}b^2c^2\left\{u\left(\dfrac{1}{c^2}-\dfrac{1}{b^2}\right)-\dfrac{v}{c^2}+\dfrac{w}{b^2}\right\}$

$=\dfrac{abc}{4R^2}b^2c^2\left\{\dfrac{w-u}{b^2}+\dfrac{u-v}{c^2}\right\}.$

Hence from symmetry we see that the above equation must reduce to

$$xb^2c^2\left\{\frac{w-u}{b^2}+\frac{u-v}{c^2}\right\}+yc^2a^2\left\{\frac{u-v}{c^2}+\frac{v-w}{a^2}\right\}$$
$$+za^2b^2\left\{\frac{v-w}{a^2}+\frac{w-u}{b^2}\right\}=0$$

which is (A).

Writing the equation of the polar circle

$$bc\cos A\,x^2+ca\cos B\,y^2+ab\cos C\,z^2=0$$

in the form

$$(xbc\cos A+yca\cos B+zab\cos C)(x+y+z)-(a^2yz+b^2zx+c^2xy)=0$$

the perpendicular in this case is found to be

$$\Sigma(z\tan B-y\tan C)\{a^2(v'+w'-u')-(u'+v'-w')ca\cos B$$
$$-(w'+u'-v')ab\cos C\}=0$$

which at once reduces to (B).

The value of the radius may be found by solving the equations (1) (2)...(6) for r^2 but as the expression thus obtained is very complicated it is preferably found by the second and third methods.

SECOND METHOD:

It is evident that if we substitute the co-ordinates of a point (x_1, y_1, z_1) in the equation of a circle given in the form

$$\Sigma ux^2+2\Sigma u'yz=0.$$

We should get k times the square of the tangent from the point where k is a constant. The square of the tangent from the centre is $-r^2$ where r is the radius.

Let (x_1, y_1, z_1) be the centre and $\frac{x-x_1}{\lambda}=\frac{y-y_1}{\mu}=\frac{z-z_1}{\nu}=r$ the equation of any line passing through it.

Substituting in the equation of the circle we have

$$f(x_1, y_1, z_1)+2r\left\{\lambda\frac{\partial f_1}{\partial x_1}+\mu\frac{\partial f_1}{\partial y_1}+\nu\frac{\partial f_1}{\partial z_1}\right\}+r^2f(\lambda, \mu, \nu)=0.$$

As the line is drawn through the centre the co-efficient of r must be zero.

$$\therefore \lambda\frac{\partial f_1}{\partial x_1}+\mu\frac{\partial f_1}{\partial y_1}+\nu\frac{\partial f_1}{\partial z_1}=0 \qquad \dots \quad (1)$$

λ, μ and ν are connected by the relations

$$\lambda+\mu+\nu=0 \qquad \dots \quad (2)$$

$$a^2\mu\nu+b^2\nu\lambda+c^2\lambda\mu=-1 \qquad \dots \quad (3)$$

Hence we have $\frac{\partial f_1}{\partial x_1}=\frac{\partial f_1}{\partial y_1}=\frac{\partial f_1}{\partial z_1}$, so that $f(x_1, y_1, z_1)=-kr^2$.

$\therefore$ $f(\lambda, \mu. \nu)=k$ where k is independent of λ, μ, ν.

Thus we have the following equations :

$$u\lambda^2+v\mu^2+w\nu^2+2u'\mu\nu+2v'\nu\lambda+2w'\lambda\mu=k$$

$$\lambda+\mu+\nu=0$$

$$a^2\mu\nu+b^2\nu\lambda+c^2\lambda\mu=1.$$

There equations hold for all values of λ, μ, ν.

Putting $\lambda=0$ we get

$$v\mu^2+2u'\mu\nu+w\nu^2=k$$

$$\mu+\nu=0$$

$$a^2\mu\nu=-1.$$

Eliminating μ and ν we get the conditions for a circle in the form

$$k=\frac{v+w-2u'}{a^2}=\frac{w+u-2v'}{b^2}=\frac{u+v-2w'}{c^2}.$$

The radius is given by $-r^2=\dfrac{f(x_1, y_1, z_1)}{k}$ where (x, y, z) is the centre.

$$\text{Now } f(\;_1, y_1, z_1)=\tfrac{1}{2}\left\{x_1\frac{\partial f_1}{\partial x_1}+y_1\frac{\partial f_1}{\partial y_1}+z_1\frac{\partial f_1}{\partial z_1}\right\}$$

$$=-\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix} \div \begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

$\therefore r^2=$the above quantity $\div k$

$$\therefore r^6=\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}^3 \div k^3 \begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^3$$

$$\therefore \text{from } k=\frac{v+w-2u'}{a^2}=\frac{w+u-2v'}{b^2}=\frac{u+v-2w'}{c^2} \qquad \dots \quad (1)$$

We have the following symmetric expression for the radius:

$$r^6=\frac{a^2b^2c^2}{(v+w-2u')\,(w+u-2v')\,(u+v-2w')}\;\frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}^3}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^3}\quad\ldots\quad(2)$$

$$=\frac{(4\Delta R)^2}{(v+w-2u')\,(w+u-2v')\,(u+v-2w')}\;\frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}^3}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^3}\quad\ldots\quad(3)$$

The equation of the nine-point circle is

$$(b^2+c^2-a^2)\,x^2+(c^2+a^2-b^2)\,y^2+(a^2+b^2-c^2)\,z^2-2a^2yz$$

$$-2b^2zx-2c^2xy=0.$$

In this case $\Pi\,(v+w-2u')=64a^2b^2c^2$.

The discriminant$=-4a^2b^2c^2$.

Writing the co-efficients of x^2, y^2, z^2 as $2\,bc\cos A$ etc. the value of the determinant in the denominator of (3) is found to be $-64\Delta^2$.

$\therefore$ the radius of the nine point circle is given by

$$r^6=\frac{(4\Delta R)^2}{64a^2b^2c^2}\cdot\frac{4^3a^6b^6c^6}{4^9\Delta^6}=\left(\frac{R}{2}\right)^6$$

where R=the circum-radius.

$\therefore r=\dfrac{R}{2}$ as is well known otherwise.

THIRD METHOD:

A simple way of finding the centre, radius, etc. of the circle is to make use of the tangential equation.

Let (x_1, y_1, z_1) be the centre and r the radius of the circle

$$ux^2+vy^2+wz^2+2u'yz+2v'zx+2w'xy=0.$$

Its tangential equation then is

$$\frac{4\Delta^2\,(lx_1+my_1+nz_1)^2}{a^2l^2+b^2m^2+c^2n^2-2mnbc\cos A-2nlca\cos B-2lmab\cos C}=r^2$$

This is the same as $Ul^2+Vm^2+Wn^2+2U'mn+2V'nl+2W'nl=0.$

Comparing the co-efficients we get

$$\frac{4\Delta^2x^2-a^2r^2}{U}=\frac{4\Delta^2y^2+b^2r^2}{V}=\frac{4\Delta^2z^2-c^2r^2}{W}$$

$$=\frac{4\Delta^2yz+r^2bc\cos A}{U'}=\frac{4\Delta^2zx+r^2ca\cos B}{V'}=\frac{4\Delta^2xy+r^2ab\cos C}{W'}$$

$$=\frac{4\Delta^2}{U+V+W+2U'+2V'+2W'}\times\{(x^2+y^2+z^2+2yz+2zx+2xy)$$

$$-r^2(a^2+b^2+c^2-2bc\cos A-2ca\cos B-2ab\cos C)\}$$

$$=-\frac{4\Delta^2}{\begin{vmatrix} u & w' & v' & 1\\ w' & v & u' & 1\\ v' & u' & w & 1\\ 1 & 1 & 1 & 0\end{vmatrix}}=-4\Delta^2k\ \text{(say)}\ \because x+y+z=1$$

and the co-efficient of $r^2=0$ identically.

Thus we get the following equations for finding the values of x, y, z, etc.:

$$x^2=\frac{a^2}{4\Delta^2}r^2-Uk \quad \ldots\ (1) \qquad yz=-\frac{bc\cos A}{4\Delta^2}r^2-U'k \quad \ldots\ (4)$$

$$y^2=\frac{b^2}{4\Delta^2}r^2-Vk \quad \ldots\ (2) \qquad zx=-\frac{ca\cos B}{4\Delta^2}r^2-V'k \quad \ldots\ (5)$$

$$z^2=\frac{c^2}{4\Delta^2}r^2-Wk \quad \ldots\ (3) \qquad xy=-\frac{ab\cos C}{4\Delta^2}r^2-W'k \quad \ldots\ (6)$$

Adding (1) (5) and (6) we get

$$x^2+zx+xy=\frac{r^2}{4\Delta^2}\{a^2-a\ (c\cos B+b\cos C)\}-(V'+W'+U)k$$
$$=-(V'+W'+U)k.$$

But $x^2+zx+xy=x(x+y+z)=x$.

Hence the co-ordinates of the centre are given by

$$x=-\frac{V'+W'+U}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}},\ y=-\frac{W'+U'+V}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}},\ z=-\frac{U'+V'+W}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}}$$

or $x=\dfrac{V'+W'+U}{U+V+W+2U'+2V'+2W'}$, etc.

If we give these values to x, y and z in $f(x,\ y,\ z)$ we get the same expression for the radius as found on page (156).

Let us now try to find the conditions for a circle.

From the equations (2), (3) and (4) we have

$$16\Delta^4k^2=\frac{(4\Delta^2y^2-b^2r^2)\ (4\Delta^2z^2-c^2r^2)-(4\Delta^2yz+r^2bc\cos A)^2}{VW-U'^2}$$

$$=\frac{-4\Delta^2r^2\ (b^2z^2+2yzbc\cos A+c^2y^2)+r^4b^2c^2\sin^2 A}{VW-U'^2}.$$

Hence writing D for the discriminant of $\Sigma ux^2+2\Sigma u'yz=0$ we get from symmetry

$$16\Delta^4k^2=\frac{-4\Delta^2r^2(c^2x^2+2cazx\cos B+a^2z^2)+r^4c^2a^2\sin^2 B}{vD}$$

$$=\frac{-4\Delta^2r^2(a^2y^2+2abxy\cos C+b^2x^2)+r^4a^2b^2\sin^2 C}{wD}$$

$$\frac{4\Delta^2r^2}{u'D}\ [\{a^2yz+zxab\cos C+xyca\cos B-x^2bc\cos A\}$$
$$+r^4a^2bc\sin B\sin C]$$

$$=\frac{-4a^2\Delta^2r^2}{(v+w-2u')D}\ \text{after eome simplification.}$$

Hence $4\Delta^2 k^2 D = \dfrac{a^2 r^2}{2u'-v-w} = \dfrac{b^2 r^2}{2v'-w-u} = \dfrac{c^2 r^2}{2w'-u-v}$ ... (A)

$$= \frac{r^2 bc \cos A}{v'+w'-u-u'} = \frac{r^2 ca \cos B}{w'+u'-v-v'} = \frac{r^2 ab \cos C}{u'+v'-w-w'}$$

Also $(v+w-2u')\ (w+u-2v')\ (v+w-2u') \cos A \cos B \cos C$

$=(u+u'-v'-w')\ (v+v'-w'-u')\ (w+w'-u'-v')$ ... (B)

The conditions for a circle are evident from (A).

A large number of elegent expressions for the radius may now be found from (A).

From the first three ratios of A we have

$$4\Delta^2 k^2 D = \frac{a^2+b^2+c^2}{2(u'+v'+w'-u-v-w)}\, r^2.$$

$$\therefore r^2 = \frac{8\Delta^2}{a^2+b^2+c^2}\,(u'+v'+w'-u-v-w)\ \frac{\begin{matrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{matrix}}{\begin{matrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{matrix}} \qquad (1)$$

The same value is obtained from the last three ratios.

Multiplying the first three ratios and transposing we get from (A)

$$r^6 = -\frac{64\Delta^6 k^6 D^3 \Pi\,(v+w-2u')}{a^2 b^2 c^2} = -\frac{4\Delta^4 k^6 D^3 \Pi\,(v+w-2u')}{R^2}$$

where R=the circum-radius

$$=\left(\frac{2\Delta^2}{R}\right)(2u'-v-w)\ (2v'-w-u)\ (2w'-u-v)\ \frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}^2}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^6} \qquad (2)$$

By (B) this result may be written as

$$r^6=\left(\frac{2\Delta^2}{R}\right)^2\frac{\{(v'+w'-u-u')\ (w'+u'-v-v')\ (u'+v'-w-w')}{\cos A\ \cos B\ \cos C}$$

$$\times\frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}^3}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^6} \quad \ldots \quad (3)$$

By adding the results in (A) we get

$$r^4=\tfrac{4}{3}\Delta^2\left[\frac{2u'-v-w}{a^2}+\frac{2v'-w-u}{b^2}+\frac{2w'-u-v}{c^2}\right]$$

$$\times\frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^2} \quad \ldots \quad (4)$$

Writing the last three ratios in the forms

$$4\Delta^2k^2D=abc\ \frac{r^2\cos A}{a(v'+w'-u-u')}=abc\ \frac{r^2\cos B}{b(w'+u'-v-v')}$$

$$=abc\ \frac{r^2\cos C}{c(u'+v'-w-w')}$$

we get

$$r^2=\frac{4\Delta^2}{3abc}\left[\frac{a}{\cos A}(v'+w'-u-u')+\frac{b}{\cos B}(w'+u'-v-v')\right.$$

$$\left.+\frac{c}{\cos C}(u'+v'-w-w')\right]\frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^2} \quad \dots \quad (5)$$

$$=\tfrac{1}{3}\frac{\Delta}{R}\left[\frac{a}{\cos A}(v'+w'-u-u')+\frac{b}{\cos B}(w'+u'-v-v')\right.$$

$$\left.+\frac{c}{\cos C}(u'+v'-w-w')\right]\frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}}{\begin{vmatrix} a & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}} \quad \dots \quad (6)$$

Before finding other expressions for the radius it will be convenient to establish the following identities:

If $ux^2+vy^2+wz^2+2u'yz+2v'zx+2w'xy=0$ represent a circle in areal co-ordinates

$$\{a^2u+b^2v+c^2w-2bcu'\cos A-2cav'\cos B-2abw'\cos C\}^3$$

$$=32\frac{\Delta^4}{R^2}(v+w-2u')(w+u-2v')(u+v-2w') \quad \dots \quad (7)$$

and

$$[a^2u+b^2v+c^2w-2bcu'\cos A-2cav'\cos B-2abw'\cos C]^2$$

$$=16\Delta^2(U+V+W+2U'+2V'+2W')$$

$$=-\frac{16\Delta^2}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}} \quad \ldots \quad (8)$$

and hence

$$(v+w-2u')^2\,(w+u-2v')^2\,(u+v-2w')^2=\left(\frac{2R^2}{\Delta}\right)^2$$

$$\times\{U+V+W+2U'+2V'+2W'\}^2$$

$$=-\left[\frac{2R^2}{\Delta}\right]^2 \frac{1}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^3} \quad \ldots \quad (9)$$

From the following two values for r^6 established already:

$$r^6=-\frac{16R^2\Delta^2}{\Pi(v+w-2u')}\cdot\frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}^3}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^3} \quad \text{and}$$

$$r^6 = -\left(\frac{2\Delta^2}{R}\right) \Pi(v+w-2u') \frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}^2}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^6}.$$

we have

$$(v+w-2u')^2 (w+u-2v')^2 (u+v-2w')^2 = -\left(\frac{2R^2}{\Delta}\right)^2$$

$$\times \begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^3$$

$$= \left(\frac{2R^2}{\Delta}\right)^2 [U+V+W+2U'+2V'+2W']^3.$$

which is the result (9).

Again

$$\frac{v+w-2u'}{a^2} = \frac{w+u-2v'}{b^2} = \frac{u+v-2w'}{c^2}$$

$$= \frac{u+u'-v'-w'}{bc\cos A} = \frac{v+v'-w'-u'}{ca\cos B} = \frac{w+w'-u'-v'}{ab\cos C} = \lambda$$

$$\therefore a^2u+b^2v+c^2w-2bcu'\cos A-2cav'\cos B-2abw'\cos C$$

$$= \frac{1}{\lambda}[\Sigma u\,(v+w-2u')-2\Sigma u'\,(u+u'-v'-w')]$$

$$= \frac{2}{\lambda}[U+V+W+2U'+2V'+2W'].$$

Cubing both sides and giving to λ the first three values we have

$$[\Sigma a^2u-2\Sigma bcu'\cos A]^3\ (v+w-2u')\ (w+u-2v')\ (u+v-2w')$$

$$=8a^2b^2c^2\ [U+V+W+2U'+2V'+2W']^3.$$

$\therefore$ from (9) we have

$$(a^2u+b^2v+c^2w-2bcu'\cos A-2cav'\cos B-2abw'\cos C)^3$$

$$=32\,\frac{\Delta^4}{R^2}\ (v+w-2u')\ (w+u-2v')\ (u+v-2w')$$

which is result (7).

If we square both sides of this identity and apply (9) we at once get

$$\{a^2v+b^2v+c^2w-2bcu'\cos A-2cav'\cos B-2abw'\cos C\}^2$$

$$=16\Delta^2(U+V+W+2U'+2V'+2W')=\frac{-16\Delta^2}{\begin{vmatrix} u & w' & v' & 1\\ w' & v & u' & 1\\ v' & u' & w & 1\\ 1 & 1 & 1 & 0\end{vmatrix}}$$

which is result (8).

From page (155) we have

$$r^2=\frac{a^2}{v+w-2u'}\ \frac{\begin{vmatrix} u & w' & v'\\ w' & v & u'\\ v' & u' & w\end{vmatrix}}{\begin{vmatrix} u & w' & v' & 1\\ w' & v & u' & 1\\ v' & u' & w & 1\\ 1 & 1 & 1 & 0\end{vmatrix}}$$

also $r^2 = -\dfrac{4\Delta^2}{a^2}(v+w-2u') \dfrac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^2}$

$\therefore$ multiplying we get

$$r^4 = -4\Delta^2 \frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}^2}{\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}^3} = 4\Delta^2 \frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}^2}{(U+V+W+2U'+2V'+2W')^3} \quad \ldots \quad (10)$$

Giving to $[U+V+W+2U'+2V'+2W']^3$ it value from (9) we get

$$r^4 = \frac{(2R)^4}{[(v+w-2u')(w+u-2v')(u+v-2w')]^2} \begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}^2$$

or

$$r = 2R \frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}^{\frac{1}{2}}}{[(2u'-v-w)(2v'-w-u)(2w'+u-v)]^{\frac{1}{2}}} \quad \ldots \quad (11)$$

Applying (8) we have

$$r^2 = 128\Delta^4 \frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}}{[a^2u+b^2v+c^2w-2bcu' \cos A-2cav' \cos B-2abw' \cos C]^3} \quad \ldots \quad (12)$$

Making use of (10) we get

$$r^2 = 4R^2 \cos A \cos B \cos C \frac{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}}{(v'+w'-u-u')(w'+u'-v-v')(u'+v'-w-w')} \quad \ldots \quad (13)$$

Several other similar expressions may be deduced by means of the identities (7), (8) and (9).

Surface waves due to a Submerged elliptic Cylinder

BY

Subodhchandra Mitra, M.Sc.

The disturbance in the flow of a stream when a cylindrical obstacle is placed in the bed was first suggested by Kelvin.[1] But he did not calculate the actual disturbance. In a recent issue of Ann Di Matematica[2] the disturbance has been calculated when a circular cylinder is placed in the bed of a stream. The object of the present paper is to find out the disturbance in the flow of a uniform stream by a submerged elliptic cylinder placed in the bed of a stream. The method adopted in this paper can be readily extended to the case of other cylinders.

Waves due to a submerged elliptic cylinder.

Let us consider the disturbance when a cylinder whose cross-section is the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is placed athwart the stream. It is supposed that the semi-major axis and the semi-minor axis of the ellipse are small compared with f the depth of the axis of the cylinder and the origin is placed in the undisturbed level of the surface vertically above the axis.

Let us take the axis of x in the direction of flow of the stream and the axis of y vertically upwards.

We know that when liquid is streaming past a fixed elliptic cylinder, the velocity potential[3] is given by

$$\phi=-cb\sqrt{\frac{a+b}{a-b}}e^{-\xi}\cos\eta-c\sqrt{a^2-b^2}\cosh\xi\cos\eta$$

where c is the general velocity of the stream and $x=d\cosh\xi\cos\eta$ and $y+f=d\sinh\xi\sin\eta$ and $d^2=a^2-b^2$. When there is disturbance in the

[1] Kelvin—Math. and Phys. Papers t IV, p. 369 (1904).

[2] Ann. Di Matematica, (3) t XXI, p. 237 (1913).
Lamb—Hydrodynamics, 4th ed., p. 402 (1916).

[3] Lamb—Hydrodynamics, p. 80.

bed, let

$$\phi=-cb\sqrt{\frac{a+b}{a-b}}e^{-\xi}\cos\eta-c\sqrt{a^2-b^2}\cosh\xi\cos\eta+X \qquad \dots \quad (1)$$

It can be proved easily that the normal velocity vanishes over the cylinder provided X is negligible in its neighbourhood.

Now transforming into polar coordinates (r,θ) we have from (1) after a little reduction,

$$\phi=-\frac{cax}{a-b}+\frac{cb}{2(a-b)}\left\{\left(r^2e^{2i\theta}-d^2\right)^{\frac{1}{2}}\right.$$

$$\left.+\left(r^2e^{-2i\theta}-d^2\right)^{\frac{1}{2}}\right\}+X,$$

where $x=r\cos\theta$, $y+f=r\sin\theta$.

Now making use of gamma-function,[1]

$$\int_0^\infty e^{-k(a+ib)}k^{n-1}dk=\frac{\Gamma(n)}{(a+ib)^n}.$$

Putting $n=-\frac{1}{2}$

$$\int_0^\infty e^{-k_1(a+ib)}\frac{dk_1}{k_1^{\frac{3}{2}}}=\Gamma(-\tfrac{1}{2})(a+ib)^{\frac{1}{2}}.$$

It we now write $y+f$ for a and $x+d$ for b, we have

$$\int_0^\infty e^{-k_1\{y+f+i(x+d)\}}\frac{dk_1}{k_1^{\frac{3}{2}}}=\Gamma(-\tfrac{1}{2})\{y+f+i(x+d)\}^{\frac{1}{2}}$$

$$=\Gamma(-\tfrac{1}{2})i^{\frac{1}{2}}\left(re^{-i\theta}+d\right)^{\frac{1}{2}}.$$

Also

$$\int_0^\infty e^{-k_2\{y+f+i(x-d)\}}\frac{dk_2}{k_2^{\frac{3}{2}}}=\Gamma(-\tfrac{1}{2})i^{\frac{1}{2}}\left(re^{-i\theta}-d\right)^{\frac{1}{2}}.$$

Therefore

$$\int_0^\infty\int_0^\infty e^{-(y+f)(k_1+k_2)-i\{k_1(x+d)+k_2(x-d)\}}\frac{dk_1dk_2}{k_1^{\frac{3}{2}}k_2^{\frac{3}{2}}}$$

$$=\{\Gamma(-\tfrac{1}{2})\}^2\,i\left(r^2e^{-2i\theta}-d^2\right)^{\frac{1}{2}}.$$

[1] Williamson—Integral Cal., p. 166.

Similarly

$$\int_0^\infty\int_0^\infty e^{-(y+f)(k_1+k_2)+i\{k_1(x+d)+k_2(x-d)\}}\,\frac{dk_1dk_2}{k_1^{\frac{3}{2}}k_2^{\frac{3}{2}}}$$

$$=\{\Gamma(-\tfrac{1}{2})\}^2\, i^3\,(r^2\,e^{2i\theta}-d^2)^{\frac{1}{2}}.$$

Thus we can write ϕ in the following way,

$$\phi=-\frac{cax}{a-b}-\frac{cb}{a-b}\int_0^\infty\int_0^\infty$$

$$\frac{e^{-(y+f)(k_1+k_2)}\sin\{(k_1+k_2)x+d(k_1-k_2)\}dk_1dk_2}{(k_1k_2)^{\frac{3}{2}}\{\Gamma(-\frac{1}{2})\}^2}+\mathrm{X}\quad\ldots\quad(2)$$

Let us assume,

$$\mathrm{X}=\int_0^\infty\int_0^\infty e^{y(k_1+k_2)}\sin\{(k_1+k_2)x+d(k_1-k_2)\}\alpha(k_1,k_2)dk_1dk_2\quad\ldots\quad(3)$$

where α is a function of k_1 and k_2 to be determined. For the equation of the free surface assumed to be steady, let us put

$$\eta_1=\int_0^\infty\int_0^\infty \beta(k_1,k_2)\cos\{(k_1+k_2)x+d(k_1-k_2)\}dk_1dk_2\quad\ldots\quad(4)$$

The conditions to be satisfied at the free surface are

$$\left[\frac{\partial\phi}{\partial y}\right]_{y=0}=\frac{cd\eta_1}{dx}\quad\ldots\quad(5)$$

Since the variable part of the pressure at the free surface will be constant,[1] *i.e.*

$$\frac{p}{\rho}=-gy-\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2,$$

or the expression

$$\frac{p}{\rho}=-g\eta_1-\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2\quad\ldots\quad(6)$$

will be independent of x provided terms of the second order are neglected.

[1] Lamb—Hydrodynamics, ibid.

From (5) we get

$$c\beta = \frac{cb}{a-b} \frac{e^{-f(k_1+k_2)}}{(k_1k_2)^{\frac{3}{2}} \{\Gamma(-\frac{1}{2})\}^2} + a \qquad \dots (7)$$

From (6) we have

$$g\beta = -\frac{c^2ab}{(a-b)^2} \frac{(k_1+k_2)e^{-f(k_1+k_2)}}{(k_1k_2)^{\frac{3}{2}}\{\Gamma(-\frac{1}{2})\}^2} + (k_1+k_2)a\frac{ca}{a-b} \qquad \dots (8)$$

From (7) and (8) we obtain

$$\beta = \frac{2b}{a-b} \frac{e^{-f(k_1+k_2)}(k_1+k_2)}{(k_1k_2)^{\frac{3}{2}} \{\Gamma(-\frac{1}{2})\}^2 \left[(k_1+k_2) - \frac{g(a-b)}{c^2a}\right]} \qquad \dots (9)$$

and eliminating β

$$a = \frac{cb}{a-b} \frac{e^{-f(k_1+k_2)} \left\{(k_1+k_2) + \frac{g(a-b)}{c^2a}\right\}}{(k_1k_2)^{\frac{3}{2}} \{\Gamma(-\frac{1}{2})\}^2} \qquad \dots (10)$$

If we write k for $\frac{g(a-b)}{c^2a}$ we have

$$\eta_1 = \frac{2b}{a-b}\int_0^\infty\int_0^\infty$$

$$\frac{e^{-f(k_1+k_2)}(k_1+k_2)\cos\{(k_1+k_2)x+d(k_1-k_2)\}dk_1dk_2}{(k_1k_2)^{\frac{3}{2}} \{\Gamma(-\frac{1}{2})\}^2 [k_1+k_2-k]} \qquad \dots (11)$$

η_1 may also be written in the form

$$\eta_1 = \frac{2b}{a-b}\Bigg[\int_0^\infty\int_0$$

$$\frac{e^{-f(k_1+k_2)}\cos\{(k_1+k_2)x+d(k_1-k_2)\}dk_1dk_2}{(k_1k_2)^{\frac{3}{2}}}$$

$$+\int_0^\infty\int_0^\infty \frac{e^{-f(k_1+k_2)}k\cos\{(k_1+k_2)x+d(k_1-k_2)\}dk_1dk_2}{(k_1k_2)^{\frac{3}{2}} [(k_1+k_2)-k]}\Bigg]$$

$$\times \frac{1}{\{\Gamma(-\frac{1}{2})\}^2} \qquad \dots (12)$$

It now remains to evaluate the integrals and determine the form of the free surface. Let us evaluate the integrals separately.

The first integral can be written in the form

$$I_1=\int_0^\infty \frac{e^{-fk_2}\cos k_2(x-d)dk_2}{k_2^{\frac{3}{2}}}\int_0^\infty \frac{e^{-fk_1}\cos k_1(x+d)}{k_1^{\frac{3}{2}}}$$

$$-\int_0^\infty \frac{e^{-fk_2}\sin k_2(x-d)dk_2}{k_2^{\frac{3}{2}}}\int_0^\infty \frac{e^{-fk_1}\sin k_1(x+d)dk_1}{k_1^{\frac{3}{2}}}$$

$$=\frac{\Gamma(-\frac{1}{2})}{\sqrt{2}}\left\{f+\{f^2+(x+d)^2\}^{\frac{1}{2}}\right\}^{\frac{1}{2}}\times\frac{\Gamma(-\frac{1}{2})}{\sqrt{2}}\left\{f+\{f^2+(x-d)^2\}^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$

$$-\frac{\left(\Gamma(-\frac{1}{2})\right)^2}{2}\left\{\{f^2+(x+d)^2\}^{\frac{1}{2}}-f\right\}^{\frac{1}{2}}\left\{\{f^2+(x-d)^2\}^{\frac{1}{2}}-f\right\}^{\frac{1}{2}}$$

$$=\frac{\{\Gamma(-\frac{1}{2})\}^2}{2}\left[\left\{f+\left(f^2+(x+d)^2\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}\left\{f+\{f^2+(x-d)^2\}^{\frac{1}{2}}\right\}^{\frac{1}{2}}\right.$$

$$\left.-\left\{\left(f^2+(x+d)^2\right)^{\frac{1}{2}}-f\right\}^{\frac{1}{2}}\left\{\left(f^2+(x-d)^2\right)^{\frac{1}{2}}-f\right\}^{\frac{1}{2}}\right].$$

The second integral is indeterminate. But it can be reduced to an integral under a sign of integration by means of the complex variable. Let us write it in the form

$$I_2=k\int_0^\infty \frac{e^{-fk_2}\cos k_2(x-d)dk_2}{k_2^{\frac{3}{2}}}\int_0^\infty \frac{e^{-fk_1}\cos k_1(x+d)dk_1}{k_1^{\frac{3}{2}}\,(k_1+k_2-k)}$$

$$-k\int_0^\infty \frac{e^{-fk_2}\sin k_2(x-d)dk_2}{k_2^{\frac{3}{2}}}\times\int_0^\infty \frac{e^{-fk_1}\sin k_1(x+d)dk_1}{k_1^{\frac{3}{2}}\,(k_1+k_2-k)}.$$

Since the integral is an even function of x, we may consider the case only when x is positive.

The properties of the integral are contained within those of the complex integral

$$\iint \frac{e^{-fy'+iy'(x-d)}dy'\cdot e^{-fy+iy(x+d)}dy}{y'^{\frac{3}{2}}\;y^{\frac{3}{2}}\;(y+y'-k)}\quad\text{where } y=u+iv,\; y'=u'+iv'.$$

Now the integral has a singularity when $y_1+y'-k=0$ where y_1 is the corresponding value of y.

The complex integral

$$=\int \frac{e^{-fy'+iy'(x-d)}dy'}{y'^{\frac{3}{2}}} \cdot \int \frac{e^{-fy+iy\,(x+d)}dy}{y^{\frac{3}{2}}\,(y-y_1)}.$$

Now it is a well-known theorem[1] in the Theory of Functions that

$$\int \frac{F(y)}{f(y)} = \frac{\pi_i F(a)}{f'(a)} \text{ where } a \text{ is a simple root of } f(y)=0$$

and the integration is taken along a circle described about a so as to exclude it.

Hence the second integral

$$=\int \frac{e^{-fy'+iy'(x-d)}dy'}{y'^{\frac{3}{2}}}\,\pi i\,\frac{e^{-fy_1+iy'(x+d)}}{y'^{\frac{3}{2}}}$$

$$=\pi i \int \frac{e^{\{-fy'+iy'(x-d)-f(k-y')+i(k-y')(x+d)\}}dy'}{y'^{\frac{3}{2}}\,(k-y')^{\frac{3}{2}}}.$$

Now the second integral is the real part of this complex integral.

Thus the second integral becomes

$$I=k\pi\int_0^\infty \frac{e^{-fk}\sin\{2k_2 d-k(x+d)\}dk_2}{k_2^{\frac{3}{2}}\,\{k-k_2\}^{\frac{3}{2}}}.$$

Therefore

$$\eta_1=\frac{b}{a-b}\Big[\ \Big\{\Big(f+\{f^2+(x+d)^2\}^{\frac{1}{2}}\Big)^{\frac{1}{2}}\Big(f+\{f^2+(x-d)^2\}^{\frac{1}{2}}\Big)^{\frac{1}{2}}\Big\}$$

$$-\Big\{\Big(f^2+(x+d)^2\Big)^{\frac{1}{2}}-f\Big\}^{\frac{1}{2}}\Big\{\Big(f^2+(x-d)^2\Big)^{\frac{1}{2}}-f\Big\}^{\frac{1}{2}}$$

$$+\frac{k\pi}{\{\Gamma(-\frac{1}{2})\}^2}\int_0^\infty \frac{e^{-fk}\sin\{2k_2 d-k(x+d)\}dk_2}{k_2^{\frac{3}{2}}\,(k-k_2)} + \text{etc.}\ \Big] \qquad (13)$$

[1] Lamb—Hydrodynamics, p. 391, p. 398.
Also Forsyth Theory of Functions. Art 24.

Waves due to a submerged circular cylinder.

From this we can deduce the form of the free surface when a circular cylinder is placed in the bed of the stream.

Let us consider the equation (13) previously obtained

$$\eta_1=\frac{b}{a-b}\Big[\int_0^\infty\int_0^\infty e^{-f(k_1+k_2)+i\{(k_1+k_2)x+d(k_1-k_2)\}}\frac{dk_1dk_2}{(k_1k_2)^{\frac{3}{2}}}$$

$$+\int_0^\infty\int_0^\infty e^{-f(k_1+k_2)-i\{(k_1+k_2)x+d(k_1-k_2)\}}$$

$$\frac{dk_1dk_2}{(k_1k_2)^{\frac{3}{2}}}\Big]\frac{1}{\{\Gamma(-\frac{1}{2})\}^2}+\text{ etc.}$$

we have seen that

$$\int_0^\infty e^{-fk_1+ik_1(x+d)}\frac{dk_1}{k_1^{\frac{3}{2}}}=\Gamma(-\tfrac{1}{2})\Big(f-i(x+d)\Big)^{\frac{1}{2}}.$$

Hence

$$\eta_1=\frac{b}{a-b}\Big[\Big(f-i(x+d)\Big)^{\frac{1}{2}}\Big(f-i(x-d)\Big)^{\frac{1}{2}}$$

$$+\Big(f+i(x+d)\Big)^{\frac{1}{2}}\Big(f+i(x-d)\Big)^{\frac{1}{2}}\Big]$$

$$=\frac{b}{a-b}\Big[(f-ix)^{\frac{1}{2}}\Big\{1-\frac{id}{f-ix}\Big\}^{\frac{1}{2}}(f-ix)^{\frac{1}{2}}\Big\{1+\frac{id}{f-ix}\Big\}^{\frac{1}{2}}$$

$$+(f+ix)^{\frac{1}{2}}\Big\{1+\frac{id}{f+ix}\Big\}^{\frac{1}{2}}\times(f+ix)^{\frac{1}{2}}\Big\{1-\frac{id}{f+ix}\Big\}^{\frac{1}{2}}\Big]+\text{etc.}$$

Now we can neglect higher powers of d above the second. To illustrate this we see that

$$\frac{d^2}{a-b}=d(a+b)=\sqrt{a^2-b^2}(a+b)=0,\text{ when } a=b.$$

The same thing holds for all the higher powers. So we can retain only the second powers of d. Therefore

$$\eta_1=\frac{b}{a-b}\left[(f-ix)\left(1+\frac{d^2}{2(f-ix)^2}\right)+(f+ix)\left(1+\frac{d^2}{2(f+ix)^2}\right)\right]+\text{etc.}$$

$$=\frac{b}{a-b}\left[2f+\tfrac{1}{2}d^2\left(\frac{1}{f-ix}+\frac{1}{f+ix}\right)\right]+\text{etc.}$$

$$=\frac{b}{a-b}\left[2f+\frac{fd^2}{f^2+x^2}\right]+\text{etc.}$$

neglecting an infinite constant

$$\eta_1=\frac{bf(a+b)}{f^2+x^2}+\text{etc.}$$

when $a=b$

$$\eta_1=\frac{2a^2f}{f^2+x^2}+\text{etc.}$$

This agrees with the form of the free surface for a circular cylinder.[1]

[1] Lamb—Hydrodynamics, p. 402, Art. 247.

AN ALGEBRAICAL IDENTITY

BY

PANDIT OUDH UPADHYAYA.

The identity $4X=Y^2-(-1)^{\frac{p-1}{2}} pz^2$, given by Gauss for the transformation of X where X represents $\frac{x^p-1}{x-1}$ is well known. There is another identity given by Eisenstein[1] for the transformation of X in the form $27X=f(U,V,W)$.

Very recently the author of this paper has shown that any number of identities can be easily obtained, and a general method of finding out these identities has been given. There it has been shown that Gauss' identity and Eisenstein's identity are only particular cases of that general theorem. The object of this paper is to find out another formula of the eleventh degree ; and it is believed that this formula has not been given by any previous writer.

Let $\eta_0, \eta_1, \eta_2, \ldots, \eta_{10}$ be the roots of the cyclotomic eleven-sectional periods and let X_1, be a polynomial of which the co-efficients are symmetric functions of the roots of $X=0$, the sum of which makes $\eta_0=0$. Similarly $X_2, X_3, \ldots, X_{11}$, are defined. [X_1, X_2, etc., have the same significance as has been given to them in Mathews' Theory of numbers.] Only that case has been considered in which X_1 may be represented in the form $U+V\eta_0$. Therefore for the case in consideration we have identically

$$X_1 = U+V\eta_0,$$

$$X_2 = U+V\eta_1,$$

$$\ldots \quad \ldots$$

$$\text{and } X_{11}=U+V\eta_{10},$$

where U and V are polynomials with integral co-efficients.

[1] Mathematische Abhandlungen (Berlin 1847) ; No. 1, Darstelung des Ausdrucks $27\,\frac{x^p-1}{x-1}$ durch eine aibische Form mit drei Variablen.

Now it is well known that

$$X = X_1 X_2 X_3 \ldots X_{11}$$

$$= (U+V\eta_0)(U+V\eta_1)(U+V\eta_2)\ldots(U+\eta V_{10})$$

$$= U^{11} + \Sigma\eta_0 U^{10}V + \Sigma\eta_0\eta_1 U^9V^2 + \Sigma\eta_0\eta_1\eta_2 U^8V^3 + \Sigma\eta_0\eta_1\eta_2\eta_3 U^7V^4$$

$$+ \Sigma\eta_0\eta_1\ldots\eta_4 U^6V^5 + \Sigma\eta_0\eta_1\ldots\eta_5 U^5V^6 + \Sigma\eta_0\eta_1\ldots\eta_6 U^4V^7$$

$$+ \Sigma\eta_0\eta_1\ldots\eta_7 U^3V^8 + \Sigma\eta_0\eta_1\ldots\eta_8 U^2V^9 + \Sigma\eta_0\eta_1\ldots\eta_9 UV^{10}$$

$$+ \eta_0\eta_1\ldots\eta_{10}V^{11}.$$

Substituting the values of these symmetric functions we find that

$$X = U^{11} - U^{10}V + aU^9V^2 - bU^8V^3 + cU^7V^4 - dU^6V^5 + eU^5V^6$$

$$-fU^4V^7 + gU^3V^8 - hU^2V^9 + iUV^{10} - jV^{11} \qquad \ldots \quad \text{(A)}$$

where a, b, c, etc., are the co-efficients of $\eta^9, \eta^8, \ldots, \eta$ and the constant term respectively in the eleven-sectional period equation.

Let us apply this theorem to the prime 23, and thus verify the formula in this case. It is known that $U = x^2 + 1$ and $V = -x$ for 23; and the period equation of the cyclotomic eleven-section for the prime 23 is

$$\eta^{11} + \eta^{10} - 10\eta^9 - 9\eta^8 + 36\eta^7 + 28\eta^6 - 56\eta^5 - 35\eta^4 + 35\eta^3$$

$$+ 15\eta^2 - 6\eta - 1 = 0.$$

$$\therefore a = -10,\ b = -9,\ c = 36,\ d = 28,\ e = -56,\ f = -35\ g = 35,\ h = 15,$$

$$i = -6 \text{ and } j = -1,$$

substituting these values in (A) we get

$$X = U^{11} - U^{10}V - 10U^9V^2 + 9U^8V^3 + 36U^7V^4 - 28U^6V^5 - 56U^5V^6$$

$$+ 35U^4V^7 + 35U^3V^8 - 15U^2V^9 - 6UV^{10} + V^{11}.$$

If we put $1 = x$ in the formula it becomes

$$23 = 2048 + 1024 - 5120 - 2304 + 4608 + 1792 - 1792 - 560$$

$$+ 280 + 60 - 12 - 1;$$

and thus we can represent 23 in the form of the eleventh degree with the help of this formula.

I should like to mention that I have received help in calculation from Pandit Shukdeo Chaubey.

Algebra of Polynomials

BY

Nripendranath Ghosh.

Chapter I.

Simple fundamental theorems.

1. Let $u_a(z)$ or simply u_a represent the rational and integral polynomial

$$a_0+a_1z+a_2z^2+a_3z^3+\ldots+a_nz^n$$

of the nth degree in z, whose coefficients $a_0, a_1, a_2 \ldots a_n$ are non-zero finite variables, mutually independent of one another. Let the first derivative of u_a (with regard to z) be represented by u'_a, the second by u''_a and generally the rth derivative ($r \not> n$) by $u_a^{(r)}$.

2. Corresponding to the polynomial u_a, let Δ_{a0} stand for the linear differential operator

$$a_1\frac{\partial}{\partial a_0}+2a_2\frac{\partial}{\partial a_1}+3a_3\frac{\partial}{\partial a_2}+\ldots+na_n\frac{\partial}{\partial a_{n-1}}$$

where $ra_r\dfrac{\partial}{\partial a_{r-1}}$ will be called the rth term of the operator and ra_r the rth coefficient. Δ_{a0} is evidently a particular type of a more general linear differential operator

$$a_1\frac{\partial}{\partial a_p}+2a_2\frac{\partial}{\partial a_{p+1}}+3a_3\frac{\partial}{\partial a_{p+2}}+\ldots+(n-p+1)a_{n-p+1}\frac{\partial}{\partial a_n}$$

where p may have any of the values $0,1,2,3\ldots n$. We shall denote this latter operator by Δ_{ap}.

3. The operators Δ_{ap} are called simple in contradistinction to another class of linear differential operators called complex. An operator will be called complex when all its coefficients involve the variable z. In the case when some of the coefficients involve the variable z and others do not, the operator will be called mixed.

4. If $\phi(W)$ be a continuous function of W then

$$\frac{d}{dz}\phi(u_n)=\Delta_{a0}\phi(u_n).$$

In proof of this theorem we observe

$$\frac{d}{dz}u_n=u'_n=(a_1+2a_2z+3a_3z^2+\ldots+na_nz^{n-1})$$

also

$$\Delta_{a0}u_n=\left(a_1\frac{\partial}{\partial a_0}+2a_2\frac{\partial}{\partial a_1}+3a_3\frac{\partial}{\partial a_2}+\ldots+na_n\frac{\partial}{\partial a_{n-1}}\right)u_n$$

$$=a_1\frac{\partial u_n}{\partial a_0}+2a_2\frac{\partial u_n}{\partial a_1}+3a_3\frac{\partial u_n}{\partial a_2}+\ldots+na_n\frac{\partial u_n}{\partial a_{n-1}}$$

$$=a_1+2a_2z+3a_3z^2+\ldots+na_nz^{n-1}$$

so that $$\frac{d}{dz}u_n=\Delta_{a0}u_n$$

$$\therefore\frac{d}{dz}\phi(u_n)=\frac{\partial\phi}{\partial u_n}u'_n=\frac{\partial\phi}{\partial u_n}\Delta_{a0}u_n=\Delta_{a0}\phi(u_n)$$

which proves the theorem.

5. The proof of the above theorem holds true as none of the coefficients $a_0, a_1, a_2\ldots a_n$ of u_n is zero, *i.e.*, as the polynomial is complete. Incomplete polynomials may be best treated by means of complex operators. In what follows, unless contrary is stated, the polynomials are always complete.

6. If $\phi(W)$ be any continuous function of W then

$$z^2\frac{d}{dz}\phi(u_n)=\left(\Delta_{a2}+na_nz\frac{\partial}{\partial a_n}\right)\phi(u_n);$$

for

$$z^2\frac{d}{dz}\phi(u_n)=\frac{\partial\phi}{\partial u_n}z^2u'_n=\frac{\partial\phi}{\partial u_n}\{a_1z^2+2a_2z^3+\ldots+(n-1)a_{n-1}z^n\}$$

$$+\frac{\partial\phi}{\partial u_n}na_nz^{n+1}$$

$$=\Delta_{a2}\phi(u_n)+na_nz\frac{\partial}{\partial a_n}\phi(u_n)$$

$$=\left(\Delta_{a2}+na_nz\frac{\partial}{\partial a_n}\right)\phi(u_n)$$

7. If $u_a, u_b, u_c \ldots$ represent a number of polynomials

$$a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n,$$

$$b_0 + b_1 z + b_2 z^2 + \ldots + b_m z^m,$$

$$c_0 + c_1 z + c_2 z^2 + \ldots + c_l z^l,$$

of n, m, lth degrees respectively and if $\Delta_{a0}, \Delta_{b0}, \Delta_{c0}, \ldots$ be operators corresponding to $u_a, u_b, u_c \ldots$ respectively, then

$$\frac{d}{dz}\phi(u_a, u_b, u_c \ldots) = (\Delta_{a0} + \Delta_{b0} + \Delta_{c0} \ldots)\phi(u_a, u_b, u_c \ldots)$$

where ϕ is a continuous function of $u_a, u_b, u_c \ldots$

In proof of this theorem we observe

$$\frac{d}{dz}u_a = u'_a = \Delta_{a0}u_a = (\Delta_{a0} + \Delta_{b0} + \Delta_{c0} \ldots)u_a$$

$$\frac{d}{dz}u_b = u'_b = \Delta_{b0}u_b = (\Delta_{a0} + \Delta_{b0} + \Delta_{c0} \ldots)u_b$$

$$\frac{d}{dz}u_c = u'_c = \Delta_{c0}u_c = (\Delta_{a0} + \Delta_{b0} + \Delta_{c0} \ldots)u_c$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\therefore \frac{d}{dz}\phi(u_a, u_b, u_c \ldots) = \frac{\partial \phi}{\partial u_a}u'_a + \frac{\partial \phi}{\partial u_b}u'_b + \frac{\partial \phi}{\partial u_c}u'_c \ldots$$

$$= (\Delta_{a0} + \Delta_{b0} + \Delta_{c0} \ldots)\phi(u_a, u_b, u_c, \ldots)$$

which proves the theorem.

The operator $(\Delta_{a0} + \Delta_{b0} + \Delta_{c0} \ldots)$ will be called a compound operator.

8. If $u_a, u_b, u_c \ldots$ represent a number of polynomials as in art 7 and if $\Delta_{a2}, \Delta_{b2}, \Delta_{c2} \ldots$ be operators corresponding to $u_a, u_b, u_c \ldots$ respectively then

$$z^2 \frac{d}{dz}\phi(u_a, u_b, u_c \ldots) = (\Delta_{a2} + \Delta_{b2} + \Delta_{c2} + \ldots + na_n z \frac{\partial}{\partial a_n}$$

$$+ mb_m z \frac{\partial}{\partial b_m} + lc_l z \frac{\partial}{\partial c_l} \ldots)\phi(u_a, u_b, u_c \ldots)$$

where ϕ is a continuous function of $u_a, u_b, u_c \ldots$

We have from Art 6

$$z^2\frac{d}{dz}u_a=\left(\Delta_{a2}+na_nz\frac{\partial}{\partial a_n}\right)u_a$$

$$z^2\frac{d}{dz}u_b=\left(\Delta_{b2}+mb_mz\frac{\partial}{\partial b_m}\right)u_b$$

$$z^2\frac{d}{dz}u_c=\left(\Delta_{c2}+lc_lz\frac{\partial}{\partial c_l}\right)u_c\ldots$$

so that

$$z^2\frac{d}{dz}\phi(u_a, u_b, u_c\ldots)$$

$$=\frac{\partial\phi}{\partial u_a}z^2\frac{d}{dz}u_a+\frac{\partial\phi}{\partial u_b}z^2\frac{d}{dz}u_b+\frac{\partial\phi}{\partial u_c}z^2\frac{d}{dz}u_c\ldots$$

$$=\frac{\partial\phi}{\partial u_a}\left(\Delta_{a2}+na_nz\frac{\partial}{\partial a_n}\right)u_a+\frac{\partial\phi}{\partial u_b}\left(\Delta_{b2}+mb_mz\frac{\partial}{\partial b_m}\right)u_b+$$

$$\frac{\partial\phi}{\partial u_c}\left(\Delta_{c2}+lc_lz\frac{\partial}{\partial c_l}\right)u_c\ldots$$

$$=\left(\Delta_{a2}+\Delta_{b2}+\Delta_{c2}+\ldots+na_nz\frac{\partial}{\partial a_n}+mb_mz\frac{\partial}{\partial b_m}\right.$$

$$\left.+lc_lz\frac{\partial}{\partial c_l}\ldots\right)\phi(u_a, u_b, u\ldots)$$

$\because$ the mixed operators are mutually independent of one another.

The operator

$$\left(\Delta_{a2}+\Delta_{b2}+\Delta_{c2}+na_nz\frac{\partial}{\partial a_n}+mb_mz\frac{\partial}{\partial b_m}+lc_lz\frac{\partial}{\partial c_l}\right)$$

is an instance of a compound mixed operator.

9. Functions of the derivatives of u_a.

We have

$$\frac{d}{dz}u_a=\Delta_{a0}u_a \qquad \ldots \quad \text{(art 4)}$$

whence $\frac{d}{dz}u'_a=\frac{d}{dz}\Delta_{a0}u_a=\Delta_{a0}\frac{d}{dz}u_a=\Delta_{a0}u'_a$

$\therefore \frac{d}{dz}u''_a=\Delta_{a0}u''_a$ and so on.

If now $\phi(u_a, u'_a, u''_a \dots u_a^{(r)})$, $r \ngtr n$

be a continuous function of u_a and its derivatives only then

$$\frac{d}{dz}\phi(u_a, u'_a \dots u_a^{(r)}) = \Delta_{a0}\phi(u_a, u'_a \dots u_a^{(r)});$$

for $\quad \dfrac{d}{dz}\phi(u_a, u'_a, u''_a \dots u_a^{(r)})$

$$= \frac{\partial \phi}{\partial u_a}\frac{du_a}{dz} + \frac{\partial \phi}{\partial u'_a}\frac{du'_a}{dz} + \frac{\partial \phi}{\partial u''_a}\frac{du''_a}{dz} + \dots + \frac{\partial \phi}{\partial u_a^{(r)}}\frac{du_a^{(r)}}{dz}$$

$$= \frac{\partial \phi}{\partial u_a}\Delta_{a0}u_a + \frac{\partial \phi}{\partial u'_a}\Delta_{a0}u'_a + \frac{\partial \phi}{\partial u''_a}\Delta_{a0}u''_a + \dots + \frac{\partial \phi}{\partial u_a^{(r)}}\Delta_{a0}u_a^{(r)}$$

$$= \Delta_{a0}\phi(u_a, u'_a, u''_a \dots u_a^{(r)})$$

which proves the theorem.

10. If $\phi(u_a, u'_a, u''_a \dots u_b, u'_b, u''_b \dots u_c, u'_c, u''_c \dots)$ be a continuous function of $u_a, u_b, u_c \dots$ and their derivatives only, then

$$\frac{d}{dz}\phi(u_a, u'_a, u''_a \dots u_b, u'_b, u''_b \dots u_c, u'_c, u''_c \dots)$$

$$= (\Delta_{a0} + \Delta_{b0} + \Delta_{c0} \dots)\phi(u_a, u'_a, u''_a \dots u_b, u'_b, u''_b \dots u_c, u'_c, u''_c \dots)$$

The proof of this theorem presents no new peculiarity. This includes theorem in Art 7 as a particular case.

11. Transformed theorems :—

Let $u_a(z)$ represent the rational and integral polynomial

$$a_0 + a_1 z + a_2 z^2 + \dots a_n z^n$$

of the nth degree in z, then it may be subjected to two distinct types of transformation giving rise to transformed polynomials.

In the first type we change the variable z to some other variable t, connected by the explicit relation $z = \psi(t)$, so that the transformed polynomial becomes $a_0 + a_1\psi(t) + a_2(\psi t)^2 + a_3(\psi t)^3 + \dots a_n(\psi t)^n$, represented by $u_a(\psi t)$.

As a result of the first type of transformation of polynomials all our previous theorems will necessarily lead to transformed theorems of the first type.

Let us take as an example, theorem in Art 4 *viz.*,

$$\frac{d}{dz}\phi(u_a)=\Delta_{a_0}\phi(u_a).$$

Put $z=\psi(t)$, then $dz=\psi'(t)dt$, so that the required transformed theorem becomes

$$\frac{1}{\psi'(t)}\cdot\frac{d}{dt}\phi\{u_a(\psi t)\}=\Delta_{a_0}\phi\{u_a(\psi t)\}$$

In the second type of transformation of u_a we change only its co-efficients $a_0 a_1 a_2 \ldots a_n$ partly or all together. Since these coefficients are non-zero finite variables mutually independent of one another, the transformation to which u_a is subjected must be such as to preserve this characteristic.

Evidently a most general form of transformation is a combination of the two types indicated here.

On the Spectra of Isotopes

BY

Panchanon Das, M.Sc.

The work of Aston and Dempster has established the fact that most elements possess two or more isotopes, which have the same atomic number and arrangement of electrons, but have nuclei of slightly different masses. The question at once arises, whether the spectra of these isotopes should be different to any extent at all. Bohr's treatment of the dynamics of the hydrogen atom with a nucleus of finite mass is easily extended to any atom and leads to the result that the Rydberg constant R, instead of being an absolute constant, would vary very slightly from element to element owing to the finite mass of the nucleus. Thus isotopes with identical electrical structure, but slightly different nuclear mass, would have their respective Rydberg-constants slightly different from each other and a separation should exist between the corresponding series-lines.

This was first put to the experimental test by Merton[1] who examined the line $\lambda = ...4058$AU of ordinary lead and compared its position with that of the corresponding line of its isotope, *viz.*, the lead derived from radioactive sources. He found the actual separation to be 0·011 AU being about 100 to 200 times as large as the theoretical value of Bohr. Quite recently MacLennan[2] in course of a study of the fine structure of the Lithium line $\lambda = 6708$AU, succeeded in resolving it into four components. As theoretically this should be doublet, he attributed this circumstance to the presence of isotopes. He computed that the actual isotopic separation was about 3 to 4 times as large as the value calculated from Bohr's theory. He finally generalised that the actual separation of the spectra of any two isotopes must be atomic number times as large as the theoretical result of Bohr. So it appears that there is something fallacious in the existing theory. Ehrenfest[3] points out that the fallacy

[1] Proc. Roy. Soc., Vol. 100.

[2] Proc. Roy. Soc., Vol. 101.

[3] Nature, June 10, 1922.

lies in applying the results of a two-body problem to an n-body problem. Bohr's original theory referred to the atoms of hydrogen and ionised helium only. The spectroscopic consequences of his theory were borne out remarkably well by facts. But when we come to an atom of a higher complexity, it becomes an n-body problem and Bohr's results cannot hold good One must investigate the joint influence of all the electrons on the motion of the nucleus in order to explain the existing discrepancy. An interesting side-light on this point was thrown by Silberstein[1] in a letter on the series-spectra of neutral helium. He finds that each of the electrons surrounding an atomic nucleus describes an orbit practically uninfluenced by the rest of electrons, and in deriving a series-formula, he makes use of the total-energy of the whole atom instead of the valency-electron only, as is usually done. The valency-electron, regarded as an isolated system, should apparently behave as non-holonomous, hence its total energy is generally a function of time. The best course is to quantise the generalised coordinates of the whole system of electrons and nucleus constituting the atom, as this last represents a conservative system. If we make certain simple assumptions, the prohibitive nature of the n-body problem gives way and a separation of variables in the Hamilton-Jacobi equation of motion can be effected.

Let M be the mass of the nucleus of an atom of any element, of which the atomic number is N and let g be the centre of mass of these N electrons. If the mass of an electron be m, the centre of mass G of the whole system consisting of electrons and nucleus divides the line Mg in the ratio of M ; Nm. Let $Gg=R_1$, and $GM=R_2$, and let the line Mg, which, we assume, lies in an invariable plane, make an angle θ with any fixed line in this plane.

Since the atom is not subject to any external forces, the point G may be regarded as fixed. We proceed to compute the kinetic energy T of the system. Since the centre of mass G is at rest the K·E of the system is equal to the K·E relative to G. Or T=K·E of M relative to G+K·E of the N electrons relative to G. Again the K·E of the N electrons relative to G=the K·E of the mass of N electrons relative to g. If we take time-average over a long period, this K·E of electrons relative to their own centre of mass may be regarded as zero. Thus,

$$T=\frac{1}{2} M(\dot{R}_2^2+R_2^2\dot{\theta}^2)+\frac{1}{2} Nm(\dot{R}_1^2+R_1^2\dot{\theta}^2),$$

where $R_1:R_2=M:Nm$.

[1] Nature, August 19, 1922.

Let us now put $R=R_1+R_2$. Then,

$$R_1=\frac{MR}{M+Nm}, \text{ and } R_2=\frac{NmR}{M+Nm}.$$

Substituting these values in T, we get,

$$T=\frac{1}{2}\cdot\frac{Nm}{1+\frac{Nm}{M}}\cdot(\dot{R}^2+R^2\dot{\theta}^2),$$

$$=\frac{1}{2}\,\mu(\dot{R}^2+R^2\dot{\theta}^2), \text{ say}$$

where

$$\mu=\frac{Nm}{1+\frac{Nm}{M}} \qquad \ldots \quad (1)$$

To find the mutual potential energy of the system we assume that all the electrons except the outermost one are grouped close together around the nucleus, so that the centre of mass g_1 of these $N-1$ electrons is very closely situated to the nucleus O, compared to the outermost electron m.

Now since $g_1g:gm=1:N-1$, it is evident that Og is approximately $\frac{1}{N}$th part of Om, if we regard Og_1 as small compared with Om. But Og was previously denoted by R. Thus, $Om=NR$, approximately.

We may regard the mutual potential energy of the $N-1$ electrons near the nucleus and the nucleus itself as remaining unaffected during a radiation, so that we may omit it from our Hamilton-Jacobian equation of motion.

Next, if we assume after Silberstein that these electrons are independent of each other, the mutual potential energy of the outer electron and the rest may be disregarded also. The only effective term in the mutual potential energy V is then the potential energy of the outer electron and the nucleus. Or,

$$V=-\frac{Ee}{Om}. \quad \text{But } E=Ne \text{ and } Om=NR.$$

Thus, $V = -\frac{e^2}{R}$ approximately.

Evidently the Hamilton-Jacobian equation of motion is

$$\frac{1}{2\mu}\left\{\left(\frac{\partial S}{\partial R}\right)^2 + \frac{1}{R^2}\left(\frac{\partial S}{\partial \theta}\right)^2\right\} - \frac{e^2}{R} = W$$

This form has been treated at length by Sommerfeld, and the total-energy W is easily seen to be

$$W = -\frac{2\pi^2 \mu e^4}{h^2} \cdot \frac{1}{n^2}$$

where n is the sum of radial and angular quantum-numbers. Now the theoretical value of Rydberg's constant R_∞ for an element with an infinite nuclear mass is

$$R_\infty = -\frac{2\pi^2 m e^4}{h^3}.$$

Hence after restoring the value of μ from (1) we get

$$W = -\frac{N \cdot R_\infty}{1 + \frac{Nm}{M}} \cdot \frac{h}{n^2}.$$

When a Quantum-transit takes place, the difference of total energy is a multiple of $h\nu$. Professor C. V. Raman suggested that since N electrons take part, this multiple might be taken to be N. Thus,

$$N \cdot h\nu = \frac{N \cdot R_\infty h}{\left(1 + \frac{Nm}{M}\right)}\left(\frac{1}{n^2} - \frac{1}{p^2}\right),$$

where p is the value of n in a different configuration.

Or, $$\frac{R}{1 + \frac{Nm}{M}}\left(\frac{1}{n^2} - \frac{1}{p^2}\right) \qquad \ldots \quad (2)$$

This is a series formula of the Balmer type. We assume that this modified Rydberg's constant will replace the usual value in the more general series-formulae also.

Now suppose that an element has got two isotopes of atomic weight M_1 and M_2 and that the frequencies of the corresponding series-lines for which the quantum-numbers are the same, are ν_1 and ν_2. Then it is easily seen after some approximations, that

$$\nu_1-\nu_2=\Delta\nu=Nm\left(\frac{1}{M_1}-\frac{1}{M_2}\right)R_\infty\left(\frac{1}{n^2}-\frac{1}{p^2}\right).$$

It is evident that this separation is N times as large as the value obtained from Bohr's formula :—

$$\nu=\frac{R_\infty}{1+\frac{m}{M}}\left(\frac{1}{n^2}-\frac{1}{p^2}\right).$$

This explains the discrepancy between observed results and values calculated from previous theories.

There are some side-issues, which call for a separate consideration. It is obvious that our modified Rydberg's constant, varies slightly from one element to another. It is not a case of steady increase or decrease with the atomic number but is of the nature of an oscillating function, as is apparent when we note that the nuclear mass M measured in terms of the mass of a hydrogen atom also varies with the atomic number and is in fact, proportional, as a first approximation, to the same. Thus N/M is a quantity of which the value oscillates between 1 and 2 for a large number of elements. If we take N/M equal to 1 for these elements then the variable Rydberg's constant becomes identical with the Rydberg constant for hydrogen. This tallies well with the remark in Fowler's Report on Series-spectra, that the series-spectra of most elements can be closely represented by means of the Rydberg constant for hydrogen.

The modified Rydberg constant was put to more rigorous test by the consideration of a particular element, helium. As the wave-lengths of the sharp series of all elements can generally be measured with the greatest accuracy, the sharp singlet series in the spectrum of helium

was chosen and the modified Rydberg constant was introduced into the Hicks formula, in order to see if any improvement could be effected. The original formula is quoted here from Fowler's Report :—

$$\nu = 1P - mS$$

$$= 27175 \cdot 17 - \frac{109722}{\left(m + 0 \cdot 862157 - \frac{0 \cdot 010908}{m}\right)}$$

where $m = 2, 3, \ldots$

We first evaluate the modified Rydberg constant for helium. Its value is R_{He}, given by :—

$$R_{He} = \frac{R_\infty}{1 + \frac{Nm}{M_{He}}} = \frac{R_\infty}{1 + \frac{2 \cdot m}{4M_H}}$$

$$= R_\infty \left(1 - \frac{m}{2M_H}\right) \text{ approximately,}$$

where M_{He} and M_H are the masses of a helium and a hydrogen nucleus respectively.

From Sommerfeld's Atom-bau, the value of R_∞ is taken to be 109737, and R_H is 109677. Hence, to calculate the value of $R_\infty \times \frac{m}{M_H}$, we have,

$$R_\infty - R_H = 109737 - 109677 = 60$$

or $$R_\infty - R_\infty \left(1 - \frac{m}{M_H}\right) = R_\infty \times \frac{m}{M_H} = 60.$$

$$\therefore R_{He} = 109737 - 30 = 109707.$$

We now replace 109722 by 109707 in the mS term of Hicks formula and introduce two other constants s, σ by way of compensation as follows :—

$$\nu=27175\cdot17-\frac{109707}{\left(m+0\cdot862157+s-0\cdot010908+\sigma\right)^2}$$

Putting $m=2$ and 3, we equate the mS term to 13445·23 and 7369·82 respectively and solve for S and σ from the two resulting equations.

The values are :—

$$s=-0\cdot000476$$

and $$\sigma=-0\cdot000486.$$

Thus the revised farmula is

$$\nu=27175\cdot17-\frac{109707}{\left(m+0\cdot861681-\frac{0\cdot010422}{m}\right)^2}\cdot$$

The series-lines arising from this formula are tabulated below :—

m	mS observed.	mS revised.	O−C($\Delta\nu$) Hicks.	O−C($\Delta\nu$) revised.
2	13445·23	13445·23	0·00	0·00
3	7369·82	7369·82	0·00	0·00
4	4646·52	4646·51	0·00	−0·01
5	3195·17	3195·21	+0·10	+0·04
6	2331·21	2331·27	+0·14	+0·06
7	1775·25	1775·59	+0·52	+0·34
8	1397·15	1397·43	+0·35	+0·21
9	1127·91	1128·33	+0·49	+0·42
10	not observed	...	...	...
11	779·93	779·85	−0·02	−0·08
12	661·48	663·28	+1·85	+1·80

The second column gives the value of mS taken from Fowler's Report, and the third the value calculated from Hicks' formula. The symbol O—C stands for the difference $\Delta\nu$ between observed and calculated values of wave-numbers.

It will be clear from a study of the last two columns that a systematic improvement is effected by our modification. It has not been possible to try it on other elements as the process is extremely laborious, but the fact that for most elements the Rydberg constant corresponds to the value for hydrogen lends strong support to our hypothesis.

On Magnetic Field due to a Thermionic Valve.

BY

K. Basu.

1. Langmuir[1] has shown that the electrons emitted by a heated metallic filament and conveyed by means of an external electric field to a concentric cylinder (anode) produce, on account of their charges an electrostatic field which tends to limit a further discharge of electrons from the heated filament. A theory of this effect has been given by Langmuir himself, which has been later on improved by von Laue.[2] They have shown that if V is the electrostatic potential in the space occupied by electrons

$$\nabla^2 V = -4\pi\rho$$

where $\rho = Ne$ is the density of electrons at any point.

Since the electrons move radially outwards from the filament to the cylinder with a definite mass velocity it is clear that in addition to an electrostatic field we shall also have an electromagnetic field. Let $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ be the vector potentials defining this field, then we have

$$\nabla^2(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = -4\pi\rho\,(u_1, u_2, u_3).$$

Now if the quantities (ρ, u_1, u_2, u_3) be known and the boundary conditions be given, $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ can be calculated at any point. From the values of $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ so determined, the magnetic field can be calculated by using the relation

$$F, G, H = \text{rot}\ \mathbf{a}.$$

The above is on the supposition that the phenomenon is perfectly steady. If the ejection of electrons be subject to fluctuations the time factor must be taken into account, we have then

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) V = -4\pi\rho,$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \mathbf{a} = -4\pi\rho u,$$

[1] Phy. Rev., Vol. II, p. 453 (1913).

[2] Laue—Jahrbuch der drahtlose Telegraphie, 1919.

where ρ, u are both functions of time. In this case the electric and magnetic fields are given by

$$\mathrm{F, G, H} = \mathrm{rot}\ \mathbf{a},$$

$$\mathrm{X, Y, Z} = -\mathrm{Grad.}\ \mathrm{V} - \frac{1}{c}\frac{\partial}{\partial t}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3).$$

2. In the present problem we need only calculate the vector-potentials given by

$$(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \int (i_x, i_y, i_z)\frac{d\Omega}{\mathrm{R}},$$

where i_x, i_y, i_z are components of current at any space-element $d\Omega$, R its distance from a *particular* point at which the *potentials* are taken. Then using cylindrical co-ordinates we find

$$\mathbf{a}_1 = \iiint \frac{i\cos\theta}{\mathrm{R}} r dr\, d\theta\, dz = \iiint \left(\frac{a}{r} i_0\right) \frac{r\cos\theta\, dr\, d\theta\, dz}{\mathrm{R}},$$

$$\mathbf{a}_2 = \iiint \frac{i\sin\theta}{\mathrm{R}} r dr\, d\theta\, dz = \iiint \left(\frac{a}{r} i_0\right) \frac{r\sin\theta\, dr\, d\theta\, dz}{\mathrm{R}},$$

$$\mathbf{a}_3 = 0,$$

where the axis of the filament (radius a) is taken as the z-axis and the central plane through the mid-point of the axis perpendicular to the generators of the cylinder (or filament) is taken as the plane $z=0$, $\theta=$ azimuth of the element $r\, dr\, d\theta\, dz$. If the co-ordinates of the particular point be (r_0, θ_0, z_0), we have

$$\mathrm{R}^2 = r_0^2 + r^2 + (z_0 - z)^2 - 2rr_0\cos(\theta - \theta_0).$$

If $(\mathbf{a}_{r_0}, \mathbf{a}_{\theta_0}, \mathbf{a}_{z_0})$ be the potentials with reference to cylindrical co-ordinates and $(\mathrm{M}_{r_0}, \mathrm{M}_{\theta_0}, \mathrm{M}_{z_0})$ the new magnetic intensity

$$\mathrm{M} = \mathrm{rot}\ \mathbf{a},$$

$$\begin{vmatrix} \mathbf{a}_{r_0} \\ \mathbf{a}_{\theta_0} \\ \mathbf{a}_{z_0} \end{vmatrix} = \begin{vmatrix} \cos\theta_0, & \sin\theta_0, & 0 \\ -\sin\theta_0, & \cos\theta_0, & 0 \\ 0, & 0, & 1 \end{vmatrix} \begin{vmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{vmatrix}$$

Thus

$$a_{r_0}=i_0a\int_{r=a}^{b}\int_{z=-l}^{l}\int_{I=0}^{2\pi}\frac{\cos I\,dr\,dz\,dI}{\{r_0^2+r^2+(z_0-z)^2-2rr_0\cos I\}^{\frac{1}{2}}},$$

$$a_{\theta_0}=i_0a\int_{r=a}^{b}\int_{z=-l}^{l}\int_{I=0}^{2\pi}\frac{\sin I\,dr\,dz\,dI}{\{r_0^2+r^2+(z_0-z)^2-2rr_0\cos I\}^{\frac{1}{2}}},$$

putting $I=\theta-\theta_0$, in the above, the limits of integration of θ and I are the same.

The values of the integrals

$$\int_0^{2\pi}\frac{\cos I\,dI}{\{r_0^2+r^2+(z_0-z)^2-2rr_0\cos I\}^{\frac{1}{2}}},$$

$$\int_0^{2\pi}\frac{\sin I\,dI}{\{r_0^2+r^2+(z_0-z)^2-2rr_0\cos I\}^{\frac{1}{2}}}$$

are respectively $\frac{2}{(r_0r)^{\frac{1}{2}}}\left[\left(\frac{2}{k}-k\right)F-\frac{2}{k}E\right]$ and zero, where $k^2=4rr_0/(r_0+r)^2+(z_0-z)^2$, and F, E, the complete elliptic integrals of the first and second kind.

Now

$$M_{r_0}=\frac{1}{r_0}\frac{\partial}{\partial\theta_0}a_{z_0}-\frac{\partial}{\partial z_0}a_{\theta_0},$$

$$M_{\theta_0}=\frac{\partial}{\partial z_0}a_{r_0}-\frac{\partial}{\partial r_0}a_{z_0},$$

$$M_{z_0}=\frac{1}{r_0}\frac{\partial}{\partial r_0}(r_0a_{\theta_0})-\frac{1}{r_0}\frac{\partial}{\partial\theta_0}a_{r_0}.$$

In the present case,

$$\left[M_{r_0},M_{\theta_0},M_{z_0}\right]=\left[0,\frac{\partial}{\partial z_0}a_{r_0},0\right]$$

Hence

$$M_{\theta_0}=i_0 a\frac{\partial}{\partial z_0}\int_{r=a}^{b}\int_{z=-l}^{l}\frac{2}{(rr_0)^{\frac{1}{2}}}\left[\left(\frac{2}{k}-k\right)F-\frac{2}{k}E\right]dr\,dz.$$

3. Now

$$F=\int_0^{\frac{\pi}{2}}\frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}=\frac{\pi}{2}\left[1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2k^4+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2k^6+\ldots\right].$$

$$E=\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\psi}\,d\psi=\frac{\pi}{2}\left[1-\left(\frac{1}{2}\right)^2k^2-\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{k^4}{3}-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\frac{k^6}{5}-\ldots\right].$$

As a first approximation, take $F=E=\frac{\pi}{2}$.

$$\therefore M_{\theta_0}=\tfrac{1}{2}\pi i_0 a\int^{b}\int_{-l}^{l}\frac{\partial}{\partial z_0}\left[\frac{2}{(rr_0)^{\frac{1}{2}}}\left\{\left(\frac{2}{k}-k\right)-\frac{2}{k}\right\}dr\,dz\right.$$

$$=\tfrac{1}{2}\pi i_0 a\int_a^{b}\int_{-l}^{l}\frac{2}{(rr_0)^{\frac{1}{2}}}\left(-\frac{\partial k}{\partial z_0}\right)dr\,dz.$$

But $-\frac{\partial k}{\partial z_0}=(z_0-z)k/(r_0+r)^2+(z_0-z)^2$, obtained by taking logarithmic differential of $k^2=4rr_0/(r_0+r)^2+(z_0-z)^2$.

Hence

$$=\tfrac{1}{2}\pi i_0 a\int^{b}\int^{l}\frac{2}{(rr_0)^{\frac{1}{2}}}\cdot\frac{(z_0-z)k}{(r_0+r)^2+(z_0-z)^2}$$

$$=\tfrac{1}{2}\,\pi i_0 a\int_a^b\int_{-l}^{l}\frac{2\,(z_0-z)\,dr\,dz}{(rr_0)^{\frac{1}{2}}\,\{(r_0+r)^2+(z_0-z)^2\}}$$

$$\frac{(4\,rr_0)^{\frac{1}{2}}}{\{(r_0+r)^2+(z_0-z)^2\}^{\frac{1}{2}}}$$

$$=2\pi i_0 a\odot,$$

$$\text{where } \odot=\int_a^b\int_{-l}^{l}\frac{(z_0-z)\,dr\,dz}{\{(r_0+r)^2+(z_0-z)^2\}^{\frac{3}{2}}}$$

$$=\int^b \{(r_0+r)^2+(z_0-z)^2 \qquad \cdot dr. \quad -l$$

$$=\int_a^b\Big[(r_0+r)^2+(z_0-l)^2\Big]^{-\frac{1}{2}}dr$$

$$-\int^b\Big[(r_0+r)^2+(z_0+l)^2\Big]^{-\frac{1}{2}}dr.$$

$$ln\,\frac{(r_0+r)^2+\sqrt{(r_0+r)^2+(z_0-l)^2}}{(r_0+r)^2+\sqrt{(r_0+r)^2+(z_0+l)^2}}$$

$$=ln\,\frac{\{(r_0+b)+\sqrt{(r_0+b)^2+(z_0-l)^2}\}}{\{r_0+b)+\sqrt{(r_0+b)^2+(z_0+l)^2}\}}$$

$$\frac{\{(r_0+a)+\sqrt{(r_0+a)^2+(z_0+l)^2}\}}{\{(r_0+a)+\sqrt{(r_0+a)^2+(z_0-l)^2}\}}$$

It can be proved very easily :

(*i*) $M_{\theta_0}(-z_0)=-M_{\theta_0}(z_0)$,

(*ii*) $M_{\theta_0}=0$, at any point in the central plane perpendicular to the axis.

(*iii*) $M_{\theta_0} = 4\Delta b z_0/r_0^3$, when r_0 is very large and $\Delta = 4\pi i_0 a l$, the total current.

(*iv*) When z_0 is larger compared to l and r_0, the logarithmic function vanishes.

4. As a second approximation put

$$F = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2\right], \quad E = \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 k^2\right].$$

$$\therefore M_{\theta_0} = \tfrac{1}{2}\pi i_0 a \int_a^b \int_{-l}^{l} \frac{2}{(rr_0)^{\frac{1}{2}}} \frac{\partial}{\partial z_0}\left[\left(\frac{2}{k} - k\right)\left(1 + \frac{1}{4}k^2\right) - \frac{2}{k}\left(1 - \frac{1}{4}k^2\right)\right] dr\, dz.$$

$$= \tfrac{1}{2}\pi i_0 a \int_a^b \int_{-l}^{l} \frac{2}{(rr_0)^{\frac{1}{2}}} \frac{\partial}{\partial z_0}\left(-\frac{1}{4}k^3\right) dr\, dz$$

$$= \tfrac{1}{2}\pi i_0 a \int_a^b \int_{-l}^{l} \frac{2}{(rr_0)^{\frac{1}{2}}} \cdot \left(-\frac{3}{4}k^2 \frac{\partial k}{\partial z_0}\right) dr\, dz$$

$$= \tfrac{3}{4}\pi i_0 a \int_a^b \int_{-l}^{l} \frac{k^2}{(rr_0)^{\frac{1}{2}}} \cdot \frac{k(z_0 - z)}{(r_0 + r)^2 + (z_0 - z)^2}\, dr\, dz$$

$$= \tfrac{3}{4}\pi i_0 a \int_a^b \int_{-l}^{l} \frac{(z_0 - z)\, dr\, dz}{(rr_0)^{\frac{1}{2}}\{(r_0 + r)^2 + (z_0 - z)^2} \cdot \frac{(4rr_0)^{\frac{3}{2}}}{\{(r_0 + r)^2 + (z_0 - z)^2\}^{\frac{3}{2}}}$$

$$= 6\pi i_0 a \int_a^b \int_{-l}^{l} \frac{rr_0 (z_0 - z)\, dr\, dz}{\{(r_0 + r)^2 + (z_0 - z)^2\}^{\frac{5}{2}}}$$

$$= 2\pi i_0 a \int^b \left| rr_0 \{(r_0 + r)^2 + (z_0 - z)^2\}^{-\frac{3}{2}} \right|_{-l}^{l} \cdot dr$$

$$=2\,\pi i_0 a \int_a^b \left[\frac{rr_0\,dr}{\{(r_0+r)^2+(z_0-l)^2\}^{\frac{3}{2}}} - \frac{rr_0\,dr}{\{(r_0+r)^2+(z_0+l)^2\}^{\frac{3}{2}}} \right]$$

Now $\displaystyle\int_a^b \frac{r\,dr}{\{(r_0+r)^2+(z_0-l)^2\}^{\frac{3}{2}}} = \int_a^b \frac{(r_0+r)\,dr}{\{(r_0+r)^2+(z_0-l)^2\}^{\frac{3}{2}}}$

$$-\int_a^b \frac{r_0\,dr}{\{(r_0+r)^2+(z_0-l)^2\}^{\frac{3}{2}}}.$$

$$\{(r_0+r)^2+(z_0-l)^2\}$$

$$-r_0 \int \frac{d\,(r_0+r)}{\{(r_0+r)^2+(z_0-l)^2\}^{\frac{3}{2}}}$$

Again $\displaystyle\int \frac{d\,(r_0+r)}{\{(r_0+r)^2+(z-l)^2\}^{\frac{3}{2}}} - \frac{r_0+r}{(z_0-l)^2\{(r_0+r)^2+(z_0-l)^2\}^{\frac{1}{2}}}$

Hence

$$M_{\theta_0} = 2\pi i_0 a r_0 \Big[-[(r_0+r)^2+(z_0-l)^2\}^{-\frac{1}{2}}$$

$$- \frac{r_0\,(r_0+r)}{(z_0-l)^2\{(r_0+r)^2+(z_0-l)^2\}^{\frac{1}{2}}}$$

$$+\{(r_0+r)^2+(z_0+l)^2\}^{-\frac{1}{2}} + \frac{r_0\,(r_0+r)}{(z_0+l)^2\{(r_0+r)^2+(z_0+l)^2\}^{\frac{1}{2}}} \Big]_{r=a}^{b}$$

$$=2\pi i_0 a r_0 \quad \frac{r_0^2+rr_0+(z_0+l)^2}{(z_0+l)^2\{(r_0+r)^2+(z_0+l)^2\}^{\frac{1}{2}}}$$

$$\left. \frac{r_0^2+rr_0+(z_0-l)^2}{(z_0-l)^2\{(r_0+r)^2+(z_0-l)^2\}^{\frac{1}{2}}} \right|_{r=a}$$

5. The value of M_{θ_0} can be expressed in terms of an integral equation involving Bessel's functions of the first type. Thus suppose

$$W=\int_0^{2\pi}\int_{-l}^{l}\int_a^b \frac{\cos I\, dr\, dz\, dI}{\sqrt{r_0^2+r^2+(z_0-z)^2-2rr_0\cos I}}$$

Then since[1] $\int_0^\infty e^{-\lambda z} J_0(\lambda R)\, d\lambda = \frac{1}{(z^2+R^2)^{\frac{1}{2}}}$, we get

$$W=\int_0^\infty\int_0^{2\pi}\int_{-l}^{l}\int_a^b e^{\pm\lambda(z_0-z)} \cos I\, J_0(\lambda R)\, dr\, dz\, dI\, d\lambda,$$

the upper or lower sign being taken according as z_0-z is negative or positive, where $R^2=r_0^2+r^2-2rr_0\cos I$.

Now[2] $J_0(\lambda R)=J_0(\lambda r_0)\, J_0(\lambda r)+2\sum_1^\infty J_s(\lambda r_0)\, J_s(\lambda r)\cos sI.$

Whence

$$W=\int_0^\infty d\lambda\int_{-l}^{l} e^{\pm\lambda(z_0-z)}dz\int_a^b dr\int_0^{2\pi}\Big\{J_0(\lambda r_0)\, J_0(\lambda r)$$

$$+2\sum_1^\infty J_s(\lambda r_0)\, J_s(\lambda r)\cos sI\Big\}\cos I\, dI.$$

Now $\int_0^{2\pi}\cos I\, dI=0$, $\int_0^{2\pi}\cos sI\cos I\, dI=\begin{cases}0, & (s\neq 1)\\ \pi, & (s=1).\end{cases}$

Hence $W=2\pi\int e^{\pm\lambda(z_0-z)}d\lambda\int_{-l}^{l} dz\int_a^b J_1(\lambda r_0)\, J_1(\lambda r)\, dr.$

[1] Gray and Mathews, 'Bessel functions.'

[2] Gray and Mathews, l.c.

Again since $J_0'(z) = -J_1(z)$, we have

$$\frac{d}{dr} J_0(\lambda r) = -\lambda J_1(\lambda r)$$

$$\therefore W = -2\pi \int_0^{\infty} e^{\pm\lambda(z_0 - z)} d\lambda \int_{-l}^{l} dz \left[J_1(\lambda r_0) . \frac{1}{\lambda} J_0(\lambda r) \right]_{r=a}^{b}$$

$$= -2\pi \int_{-l}^{l} \int_0^{\infty} e^{\pm\lambda(z_0 - z)} J_1(\lambda r_0) \{ J_0(\lambda b) - J_0(\lambda a) \} \frac{dz\, d\lambda}{\lambda} .$$

$$= -2\pi \int^{\infty} e^{\pm\lambda z_0} (e^{\lambda l} - e^{-\lambda l}) J_1(\lambda r_0) J_0(\lambda b) \frac{d\lambda}{\lambda^2} .$$

$$+ 2\pi \int_0^{\infty} e^{\pm\lambda z_0} (e^{\lambda l} - e^{-\lambda l}) J_1(\lambda r_0) J_0(\lambda a) \frac{d\lambda}{\lambda^2} ,$$

Now $M_{\theta_0} = i_0 a \dfrac{\partial}{\partial z_0} W$

$$= \mp 2\pi i_0 a \left[\int e^{\pm\lambda z_0} (e^{\lambda l} - e^{-\lambda l}) J_1(\lambda r_0) J_0(\lambda b) \frac{d\lambda}{\lambda} \right.$$

$$\left. - \int^{\infty} e^{\pm\lambda z_0} (e^{\lambda l} - e^{-\lambda l}) J_1(\lambda r_0) J_0(\lambda a) \frac{d\lambda}{\lambda} \right] .$$

The *minus* sign before the right-hand side corresponds to $+\lambda z_0$ and *plus* sign corresponds to $-\lambda z_0$; $+\lambda z_0$ is taken for $z_0 < l$ and $-\lambda z_0$ for $z_0 > l$.

We find

$$M_{\theta_0} = -2\pi i_0 a \int_0^\infty e^{\pm\lambda(z_0+l)} J_1(\lambda r_0) J_0(\lambda b) \frac{d\lambda}{\lambda}$$

$$+ 2\pi i_0 a \int_0^\infty e^{\pm\lambda(z_0-l)} J_1(\lambda r_0) J_0(\lambda b) \frac{d\lambda}{\lambda}$$

$$+ 2\pi i_0 a \int_0^\infty e^{\pm\lambda(z_0+l)} J_1(\lambda r_0) J_0(\lambda a) \frac{d\lambda}{\lambda}$$

$$- 2\pi i_0 a \int_0^\infty e^{\pm\lambda(z_0-l)} \cdot J_1(\lambda r_0) J_0(\lambda a) \frac{d\lambda}{\lambda}.$$

As before the $\pm\lambda$ is taken according as $z_0 <$ or $> l$.

These are well-known integrals. We have met with this type of integrals in Hydrodynamics as the expression for velocity potential for sources distributed with uniform density over the plane area contained by a circle,[1] also from analogy the gravitational potential produced at any outside-point due to a thin disc of matter of uniform surface density has got the same value.[2]

Lastly I wish to express my thanks to Prof. M. N. Saha, D.Sc., who suggested the problem to me for his encouragement in this direction.

[1] Lamb's 'Hydrodynamics,' 4th edition, p. 131.
[2] Gray. 'Phil. Mag., August, 1919,' p. 203.

Ripples of Finite Amplitude on a Viscous Liquid

BY

J. C. Kamesvara Rav, M.Sc.

In a paper published in this bulletin,[1] I showed the change of form of waves of finite amplitude, as the wave length increases from those of short ripples to those of large gravity waves, without taking into consideration the effect of viscosity, which however, is not a negligible factor, as it tends to damp the amplitude, which in its turn affects the form of the wave.

The effect of viscosity on waves on the surface of liquids first received the attention of Stokes,[2] who by employing the dissipation function found the modulus of decay to vary as ν^{-1}. Tait[3] discussed the effect on short ripples and showed that it is more prominent in the case of shorter ripples. Harrison[4] found for superposed liquids, the modulus of decay to vary as $\nu^{-\frac{1}{2}}$. Basset[5] extended the case to liquids of finite depth. Recently Watson[6] made some experimental investigations to find the viscosity of liquids by taking observations of the decay of the amplitude of surface waves. He, however worked only with small amplitudes. Taking the effect of finiteness of the amplitude, we can proceed as follows—

The motion is supposed to be confined to two dimensions. The axis of X is drawn in the direction of propagation and the axis of Y is drawn vertically upwards.

[1] Bull. of the Cal. Math. Soc. Vol. XI, p. 173 (1920). See also Proc. Ind. Ass. f. Cult. of Science. Vol. VI, p. 175 (1921).

[2] Camb. Trans. t. ix, p. 8, 1855 or papers Vol. III, p. 1.

[3] Proc. Roy. Soc. Edin. Vol. XVII, p. 110 (1890) or Scientific papers Vol. II. p. 313.

[4] Proc. Lond. Math. Soc. (2) Vol. VI, p. 396 and Vol. VII, p. 107 (1908).

[5] Hydrodynamics, Vol. II, §§. 520-522 (1888).

[6] Phys. Rev. Vol. VII, p. 226 (1916).

The well known equations for the motion of a viscous fluid are

$$\frac{Du}{Dt}=X-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u$$

$$\frac{Dv}{Dt}=Y-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v$$

In the present case these reduce to

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u \qquad \ldots \quad 1)$$

$$\frac{\partial v}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 v \qquad \ldots \quad (2)$$

The equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad \ldots \quad 3)$$

These equations are satisfied by

$$\left.\begin{aligned} u&=-\frac{\partial \phi}{\partial x}-\frac{\partial \psi}{\partial y} \\ v&=-\frac{\partial \phi}{\partial y}+\frac{\partial \psi}{\partial x}, \end{aligned}\right\} \qquad \ldots \quad (4)$$

and the pressure equation at the surface

$$\frac{p}{\rho}=\frac{\partial \phi}{\partial t}-gy-\frac{1}{2}(u^2+v^2) \qquad \ldots \quad (5)$$

provided that

$$\nabla_1^2\phi=0, \text{ and } \frac{\partial \psi}{\partial t}=\nu\nabla^2\psi, \qquad \ldots \quad (6)$$

where $\nabla_1^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

The condition of 'no motion' at the bottom is given by

$$-\frac{\partial \phi}{\partial y}+\frac{\partial \psi}{\partial x}=0, \text{ when } y=-h. \qquad \ldots \quad (7)$$

Solutions of (6) and (7) are given by

$$\phi = \sum_{s=n}^{s=1} A_s e^{sat} \cosh sk(y+h)\cos skx$$

and

$$\psi = \sum^{s=1} B_s e^{sat} \sinh sm(y+h)\sin skx$$

where

$$m^2 = k^2 + \frac{a}{\nu}. \qquad \ldots \quad (8)$$

The boundary conditions supply enough equations to determine the constants $A_2, A_3 .. B_2, B_3 ...$ and a, A_1 and B_1 (which are related to each other) remaining arbitrary.

For our purpose it is sufficient to take only three terms, and we have

$$\phi = A_1 e^{at} \cosh k(y+h)\cos kx + A_2 e^{2at} \cosh 2k(y+h)\cos 2kx$$
$$+ A_3 e^{3at} \cosh 3k(y+h)\cos 3kx, \qquad \ldots \quad (9)$$

and $$\psi = B_1 e^{at} \sinh m(y+h)\sin kx + B_2 e^{2at} \sinh 2m(y+h)\sin 2kx$$
$$+ B_3 e^{3at} \sinh 3m(y+h)\sin 3kx. \qquad \ldots \quad (10)$$

Substituting these values in equations (4), we get

$$u = \{kA_1 \cosh k(y+h) - mB_1 \cosh m(y+h)\}e^{at} \sin kx$$
$$+ 2\{kA_2 \cosh 2k(y+h) - mB_2 \cosh 2m(y+h)\}e^{2at} \sin 2kx$$
$$+ 3\{kA_3 \cosh 3k(y+h) - mB_3 \cosh 3m(y+h)\}e^{3at} \sin 3kx. \qquad (11)$$

$$v = -k\{A_1 \sinh k(y+h) - B_1 \sinh m(y+h)\}e^{at} \cos kx$$
$$-2k\{A_2 \sinh 2k(y+h) - B_2 \sinh 2m(y+h)\}e^{2at} \cos 2kx$$
$$-3k\{A_3 \sinh 3k(y+h) - B_3 \sinh 3m(y+h)\}e^{3at} \cos 3kx. \quad \ldots \quad (12)$$

If η denotes the elevation of the free surface and if the origin be taken in the undisturbed level of the liquid we have

$$v = \frac{\partial \eta}{\partial t},$$

$$\eta = -\frac{k}{a}\Big[(A_1 \sinh kh - B_1 \sinh mh)e^{at} \cos kx$$
$$+ (A_2 \sinh 2kh - B_2 \sinh 2mh)e^{2at} \cos 2kx$$
$$+ (A_3 \sinh 3kh - B_3 \sinh 3mh)e^{3at} \cos 3kx\Big]. \qquad \ldots \quad (13)$$

The stress conditions at the surface ($\eta=0$), give

$$p_{yy}=\mathrm{T}\frac{\partial^2\eta}{\partial x^2}, \qquad \ldots \quad (14)$$

and

$$p_{xy}=0, \qquad \ldots \quad (15)$$

where T is the surface tension of the liquid. The curvature is supposed to be small.

But

$$p_{yy}=-p+2\mu\frac{\partial v}{\partial y} \qquad \ldots \quad (16)$$

$$p_{xy}=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right), \qquad \ldots \quad (17)$$

where $\mu=\rho\nu$.

Equations (17) and (16) together give the values of the constants $A_2, A_3 \ldots, B_2, B_3$ etc. and a. Equation (17), gives on substitution

$$\{2A_1k^2\sinh kh-B_1(k^2+m^2)\sinh mh\}e^{at}\sin kx+4\{2A_2k^2\sinh 2kh$$
$$-B_2(k^2+m^2)\sinh 2mh\}e^{2at}\sin 2kx+9\{2A_3k^2\sinh 3kh$$
$$-B_3(k^2+m^2)\sinh 3mh)\}e^{3at}\sin 3kx=0.$$

Equating the coefficients of $\sin kx$, $\sin 2kx$ and $\sin 3kx$ to zero, we get.

$$\frac{B_1}{A_1}=\frac{2k^2\sinh kh}{(k^2+m^2)\sinh mh} \qquad \ldots \quad (18)$$

$$\frac{B_2}{A_2}=\frac{2k^2\sinh 2kh}{(k^2+m^2)\sinh 2mh} \qquad \ldots \quad (19)$$

$$\frac{B_3}{A_3}=\frac{2k^2\sinh 3kh}{(k^2+m^2)\sinh 3mh} \qquad \ldots \quad (20)$$

Equations (16) and (14) together give, after substituting the values of η, u and v from (12), (13) and (14)

$$0=\frac{k}{a}\left(g+\frac{\mathrm{T}}{\rho}k^2\right)\left(A_1\sinh kh-B_1\sinh mh\right)e^{at}\cos kx$$

$$+\frac{k}{a}\left(g+\frac{4\mathrm{T}}{\rho}k^2\right)\left(A_2\sinh 2kh-B_2\sinh 2mh\right)e^{2at}\cos 2kx$$

$$+\frac{k}{a}\left(g+\frac{9\mathrm{T}}{\rho}k^2\right)\left(A_3\sinh 3kh-B_3\sinh 3mh\right)e^{3at}\cos 3kx$$

$$+A_1 a e^{at}\cos kx\cosh k(\eta+h)+2A_2 a e^{2at}\cos 2kx\cosh 2k(\eta+h)$$

$$+3A_3 a e^{3at}\cos 3kx\cosh 3k(\eta+h)+2\nu k\{A_1 k\cosh k(\eta+h)$$

$$-mB_1\cosh m(\eta+h)\}e^{at}\cos kx+4\nu k\{A_2 k\cosh 2k(\eta+h)$$

$$-mB_2\cosh 2m(\eta+h)\}e^{2at}\cos 2kx+6\nu k\{A_3 k\cosh 3k(\eta+h)$$

$$-mB_3\cosh 3m(y+h)\}e^{3at}\cos 3kx$$

$$-\frac{1}{2}\Big[\{kA_1\cosh k(\eta+h)-mB_1\cosh m(\eta+h)\}e^{at}\sin kx$$

$$+2\{kA_2\cosh 2k(\eta+h)-mB_2\cosh 2m(\eta+h)\}e^{2at}\sin 2kx$$

$$+3\{kA_3\cosh 3k(\eta+h)-mB_3\cosh 3m(\eta+h)\}e^{3at}\sin 3kx\Big]^2$$

$$-\frac{k^2}{2}\Big[\{A_1\sinh k(\eta+h)-B_1\sinh m(\eta+h)\}e^{at}\cos kx$$

$$+2\{A_2\sinh 2k(\eta+h)-B_2\sinh 2m(\eta+h)\}e^{2at}\cos 2kx$$

$$+3\{A_3\sinh 3k(\eta+h)-B_3\sinh 3m(\eta+h)\}e^{3at}\cos 3kx\Big]^2.$$

Expanding the hyperbolic functions in powers of $(\eta+h)$ and substituting the value of η given by (13) in the above equation and equating the coefficients of $\cos kx$, $\cos 2kx$ and $\cos 3kx$ to zero, we get,

$$\frac{k}{a}\left(g+\frac{Tk^2}{\rho}\right)(A_1\sinh kh-B_1\sinh mh)+A_1 a\cosh kh$$
$$+2\nu k(A_1 k\cosh kh-B_1 m\cosh mh)=0. \qquad \ldots \quad (21)$$

$$\frac{k}{a}\left(g+\frac{4T}{\rho}k^2\right)(A_2\sinh 2kh-B_2\sinh 2mh)+2A_2 a\cosh 2kh$$

$$+\frac{1}{2}k^3A_1\sinh kh(A_1\sinh kh-B_1\sinh mh)+4\nu k(A_2 k\cosh 2kh$$

$$-B_2 m\cosh 2mh)+\nu k^5/a\cdot A_1(A_1\sinh kh-B_1\sinh mh)\sinh kh$$

$$+\nu/a\cdot k^3m^2B_1(A_1\sinh kh-B_1\sinh mh)\sinh mh$$

$$-\frac{1}{2}(kA_1\cosh kh-mB_1\cosh mh)^2-\frac{k^2}{2}(A_1\sinh kh-B_1\sinh mh)^2=0, \quad (2?$$

$$\frac{k}{a}\left(g+\frac{9Tk^2}{\rho}\right)(A_3\sinh 3kh-B_3\sinh 3mh)+3A_3 a\cosh 3kh$$

$$+6\nu k(A_3 k\cosh 3kh-B_3 m\cosh 3mh)$$

$$+\frac{1}{8}\,A_1\frac{k^4}{a}(A_1\sinh kh-B_1\sinh mh)^2\cosh kh$$

$$-\frac{1}{2}\,A_1^2k(A_2\sinh 2kh-B_2\sinh 2mh)\sinh kh$$

$$-2A_2k^3(A_1\sinh kh-B_1\sinh mh)\sinh 2kh$$

$$+2\nu k\Big[\,\frac{A_1}{8}\,\frac{k^5}{a^2}(A_1\sinh kh-B_1\sinh mh)^2\cosh kh$$

$$-\frac{1}{2}\,A_1\frac{k^3}{a}(A_2\sinh 2kh-B_2\sinh 2mh)\sinh kh$$

$$-2A_2\,\frac{k^3}{a}(A_1\sinh kh-B_1\sinh mh)\,\sinh 2kh\,\Big]$$

$$+2\nu k\Big[-\,\frac{B_1}{8}\,\frac{k^2m^3}{a^2}\,(A_1\sinh kh-B_1\sinh mh)^2\cosh mh$$

$$+\frac{1}{2}\,B_1\frac{km^2}{a}\,(A_2\sinh 2kh-B_2\sinh 2mh)\sinh mh$$

$$+2B_2\frac{km^2}{a}\,(A_1\sinh kh-B_1\sinh mh)\sinh 2mh\,\Big]$$

$$-(kA_1\cosh kh-mB_1\cosh mh)(kA_2\cosh 2kh-mB_2\cosh 2mh)$$

$$-k^2(A_1\sinh kh-B_1\sinh mh)(A_2\sinh 2kh-B_2\sinh 2mh)=0. \quad \dots \quad (23)$$

Substituting the value of B_1/A_1 as given by (18) in equation (21), we get with the help of (8)

$$-\frac{k}{a}\left(g+\frac{Tk^2}{\rho}\right)\frac{\sinh kh}{2k^2\nu+a}-(a+2\nu k^2)\cosh kh=0,$$

neglecting squares of ν.

The above equation gives

$$a=-2\nu k^2\mp ik(g/k+Tk/\rho)^{\frac{1}{2}}$$

$$=-2\nu k^2\mp ikc \qquad \dots \quad (24)$$

where c is written for $(g/k+Tk/\rho)$, the velocity of the wave in absence of friction.

Equation (22) becomes on the substitution of the values of B_1 and B_2 from (18) and (19) and with the help of (8)

$$\frac{A_2k}{a}\left(g+\frac{4Tk^2}{\rho}\right)\left(\frac{a}{2k^2\nu+a}\right)\sinh 2kh+2A_2a\cosh 2kh$$

$$+\frac{1}{2}k^3A_1^2\frac{a}{2k^2\nu+a}\cdot\sinh^2 kh+4k\nu A_2\left\{k\cosh 2kh\right.$$

$$\left.-\frac{2k^2\nu}{2k^2\nu+a}\cdot\frac{\sinh 2kh}{\tanh 2mh}\right\}+\frac{\nu k^5A_1^2}{2k^2\nu+a}\sinh^2 kh$$

$$\frac{2k^5m^2\nu^2A_1^2}{(2k^2\nu+a)^2}\sinh^2 kh-\frac{1}{2}k^3A_1^2\cosh 2kh$$

$$-\frac{1}{2}A_1^2\frac{2k^2\nu\sinh kh}{(2k^2\nu+a)\sinh mh}\left(k^2\cosh 2mh+\frac{a}{\nu}\cosh^2 mh\right)=0.$$

From this we get, neglecting squares of ν

$$A_2=-\frac{\frac{1}{2}k^3A_1^2 a\sinh^2 kh+\nu k^5A_1^2\sinh^2 kh+}{k\left(g+\frac{4Tk^2}{\rho}\right)\sinh 2kh+2a\cosh 2kh(2k^2\nu+a)-4k\nu[k(2k^2\nu+a)\cosh 2kh-2\nu k^2\sinh 2kh\coth 2mh]} \quad \ldots \quad (25)$$

Equation (23) becomes

$$\frac{A_3k}{a}\left(g+\frac{9Tk^2}{\rho}\right)\frac{a\cdot\sinh 3kh}{2k^2\nu+a}+3A_3a\cosh 3kh+6A_3\nu k\left[k\cosh 3k\right.$$

$$\left.\frac{2mk^2\nu}{2k^2\nu+a}\cdot\frac{\sinh 3kh}{\coth 3mh}\right]+\frac{1}{8}A_1^3\frac{k^4\cdot a}{(2k^2\nu+a)^2}\sinh^2 kh\cosh kh$$

$$-\frac{5}{2}A_1A_2\frac{ak^3}{2k^2\nu+a}\sinh 2kh\sinh kh.$$

$$+2\nu k\left\{\frac{2k^2\nu}{2k^2\nu+a}\sinh kh\coth mh\left[\frac{A_1^3}{8}\frac{k^5}{(2k^2\nu+a)^2}\sinh^2 kh\right.\right.$$

$$\left.-\frac{1}{2}A_1A_2\frac{km^2}{2k^2\nu+a}\sinh 2kh\tanh mh\right]$$

$$\left.+2A_1A_2km^2\frac{2k^2\nu}{2k^2\nu+a}\sinh 2kh\sinh kh\right\}.$$

$$+A_1A_2\left(k\cosh kh-\frac{2mk^2\nu}{2k^2\nu+a}\sinh kh\coth mh\right)\left(k\cosh 2kh\right.$$

$$\left.-\frac{2mk^2\nu}{2k^2\nu+a}\sinh 2kh\tanh 2mh\right)-A_1A_2k^2\cdot\frac{a^2}{(2k^2\nu+a)^2}$$

$$\sinh kh\sinh 2kh+\text{etc.}=0.$$

This, on further simplification becomes

$$A_3 k\left(g+\frac{9Tk^2}{\rho}\right)\frac{\sinh 3kh}{2k^2\nu+a}+3A_3 a\cosh 3kh+6A_3\nu k\left[k\cosh 3kh\right.$$

$$\left.-\frac{2mk^2\nu}{2k^2\nu+a}\,\frac{\sinh 3kh}{\coth 3mh}\right]+\frac{A_1^3}{4}\,\frac{k^4 a}{2k^2\nu+a}\sinh^2 kh\cosh kh$$

$$-\frac{5}{2}A_1A_2\sinh kh\sinh 2kh=0.$$

From this,

$$A_3=\frac{\frac{5}{2}A_1A_2k(2k^2\nu+a)^2\sinh kh\sinh 2kh-\frac{A_1^3}{4}k^4c^2\sinh^2 kh\cosh kh}{k\left(g+\frac{9Tk^2}{\rho}\right)\sinh 3kh+3a(2k^2\nu+a)\cosh 3kh+6\nu k[(2k^2\nu+a)\cosh 3kh-2mk^2\nu\sinh 3kh\tanh 3mh]} \quad \ldots \quad (26)$$

With the help of relations (18), (19) and (20), the equation of the wave surface can be written as

$$\eta=-\frac{k}{2k^2\nu+a}\left[A_1e^{at}\sinh kh\cos kx+A_2e^{2at}\sinh 2kh\cos 2kx\right.$$

$$\left.+A_3e^{3at}\sinh 3kh\cos 3kx\right].$$

Substituting the value of a, given by (24), we get

$$\eta=-\frac{k}{2k^2\nu+a}\left[A_1e^{-2k^2\nu t}e^{ikct}\sinh kh\cos kx\right.$$

$$\left.+A_2e^{-4k^2\nu t}e^{2ikct}\sinh 2kh\cos 2kx+A_3e^{-6k^2\nu t}e^{3ikct}\sinh 3kh\cos 3kx\right]$$

Substituting the values of A_2 and A_3 given by (25) and (26) we get,

$$\eta=-\frac{k}{2k^2\nu+a}\left[A_1e^{-2k^2\nu t}e^{ikct}\sinh kh\right.$$

$$-\frac{\frac{1}{2}k^3A_1^2a\sinh^2 kh+\nu k^5A_1^2\sinh^2 kh+}{k\left(g+\frac{4Tk^2}{\rho}\right)\sinh 2kh+2(2k^2\nu+a)a\cosh 2kh}$$

$$\times e^{-4k^2\nu t}e^{2ikct}\sinh 2kh$$

$$+\frac{\frac{5k^2}{2}A_1A_2(2k^2\nu+a)^2\sinh kh\sinh 2kh-\frac{A_1^3}{4}k^4c^2\sinh^2 kh\cosh kh}{k\left(g+\frac{4Tk^2}{\rho}\right)\sinh 3kh+3(2k^2\nu+a)a\cosh 3kh}$$

Putting $\quad -\dfrac{kA_1 \sinh kh}{2k^2\nu+\alpha}=a,$ we get

$$\eta=ae^{-2\nu k^2 t}e^{ikct}\cos kx$$

$$+\frac{\left(\frac{1}{2}k^3c^2+\nu k^3c^2\right)a^2\cos 2kx}{k\left(g+\dfrac{4Tk^2}{\rho}\right)\sinh 2kh-2kac\cosh 2kh}\cdot e^{-4\nu k^2 t}e^{2ikct}\sinh 2kh$$

$$+\frac{\frac{5}{2}aA_2c^2k^2\sinh 2kh-\frac{a^3}{8}k^4c^2\coth kh}{\left(g+\dfrac{9Tk^2}{\rho}\right)\sinh 3kh-3kcac\cosh 3kh}e^{-6\nu k^2 t}e^{3ikct}\sinh 3kh\cos 3kx.$$

In real quantities this can be written as

$$\eta=ae^{-2\nu k^2 t}\cos kct\cos kx$$

$$+\frac{\left(\frac{1}{2}k^2c^2+\nu k^2c^2\right)e^{-4\nu k^2 t}\sinh 2kh\cos 2kct\cos 2kx}{\left(g+\dfrac{4Tk^2}{\rho}\right)\sinh 2h-2kc^2\cosh 2kh}$$

$$+\frac{\frac{5}{2}aA_2k^2c^2\sinh 2kh-\frac{a}{8}k^4c^2\coth kh}{\left(g+\dfrac{9Tk^2}{\rho}\right)\sinh 3kh-3kc^2\cosh 3kh}$$

$$\times e^{-6\nu k^2 t}\sinh 3kh\cos 3kct\cos 3kx. \qquad \ldots \quad (27)$$

For infinite depth, this becomes finally.

$$\eta=ae^{-2\nu k^2 t}\cos kct\cos kx$$

$$+\frac{\left(\frac{1}{2}k^2c^2+\nu k^2c^2\right)a^2}{\left(g+\dfrac{4Tk^2}{\rho}\right)-2kc^2}e^{-4\nu k^2 t}\cos 2kct\cos 2kx$$

$$+\frac{\frac{5}{2}aa'k^2c^2-\frac{a^3}{8}k^4c^2}{\left(g+\dfrac{9Tk^2}{\rho}\right)-3kc^2}e^{-6\nu k^2 t}\cos 3kct\cos 3kx \qquad \ldots \quad (28)$$

where a' is given by

$$\frac{\frac{1}{2}k^2c^2+\nu k^2c^2}{\left(g+\frac{4Tk^2}{\rho}\right)-2kc^2}.$$

Equation (28) shows that the effect of viscosity is greater on the second term than on the first ; hence the viscosity has an effect on the form of the wave similar to that of divergence. The above solution also shows that the viscosity effects the form of small ripples more than those of large gravity waves. A similar affect has also been observed experimentally. In the case of long waves, the division of waves into two crests, extends over many wave lengths, while in the case of short ripples no division is noticed. The author hopes to make a more detailed experimental verification in the near future.

Geometrical Investigations on the Correspondences between a Right-Angled Triangle, a Three-Right-Angled Quadrilateral and a Rectangular Pentagon in Hyperbolic Geometry.

BY

S. Mukhopadhyaya.

Theorem 1.

ABC is a triangle right-angled at C in a given hyperbolic plane. AU is parallel to CB and DV is parallel to AB, where D is a point in AC produced and the angle ADV is a right angle. EF is the common perpendicular to AU and DV, meeting AU in E and DV in F.

Then is AE equal to AB and DF equal to CB. See Fig. 1.

Proof.

With a view to perspicuity the proof will be divided into several distinct parts.

(1)

Let G be the middle point of CD and H of EF. Produce CB to X and AB to Y. Join GH and produce it to Z.

Then GZ is parallel to CX. See Fig. 1.

As G is the middle point of CD which is common perpendicular to CX and DV, a parallel GW to VD will in the opposite sense WG be parallel to CX and therefore to AU. Hence WG passes through the middle point H of the common perpendicular EF to AU and DV.

(2)

From AU cut off AE′ equal to AB and from DV cut off DF′ equal to CB.

Then the angles AE′F′ and DF′E′ are equal. See Fig. 2.

Let p denote the line which is common parallel[1] to CX and AY and consequently also common parallel to AU and DV. Join BE′ and BF′. The bisector of the angle BAE′ which bisects BE′ at right angles is perpendicular to p. The perpendicular bisector of CD which bisects BF′ at right angles is also perpendicular to p. It follows from Bolyai's Theorem that the perpendicular to E′F′ through its middle point H′ is also perpendicular to p. It follows from considerations of symmetry that the angles AE′F′ and DF′E′ are equal.

(3)

Produce AD to K making DK equal to AC, so that G the middle point of CD is also the middle point of AK. Join KF′. From the congruence of the triangles KDF′ and ACB, KF′ is equal to AB and parallel to CX.

Then GH′ joining the middle points of AK and E′F′ is parallel to CX. See Fig. 3.

Through G draw the parallel GZ to AU. Produce KF′ to T. Then GZ is also parallel to CX and KT. Draw perpendiculars AM, E′N, KP, F′Q on GZ. Then AM is equal to KP because AG is equal to GK. Consequently the angle of parallelism MAU is equal to the angle of parallelism PKT. Also AE′ is equal to KF′ as each of them is equal to AB. Therefore the figures AMNE′ and KPQF′, are congruent, so that E′N is equal to F′Q. Hence the line GZ passes through H′ the middle point of E′F′. See Fig. 3.

(4)

It follows from (1), (2), (3) that EF must coincide with E′F′.

For if EF do not coincide with E′F′, it follows from (2) that HH′ is perpendicular to EF. But it follows from (1) and (3) that HH′ is parallel to AE and therefore cannot be perpendicular to EF.

Thus Theorem 1 is completely proved.

COROLLARY 1, THEOREM 1.

If a and b denote two sides of a right-angled triangle and c the hypotenuse and if λ, μ denote the angles opposite the sides a, b, then a three-right-angled quadrilateral can always be constructed of which the fourth angle is β and the sides reckoned in order from this side are l, a, m', c,

[1] Hilbert in his *Grundlagen* gives an elegant construction of the common parallel which is independent of the *Postulate of Archimedes*, reproduced by Carslaw in his Non-Euclidean Geometry, p. 55.

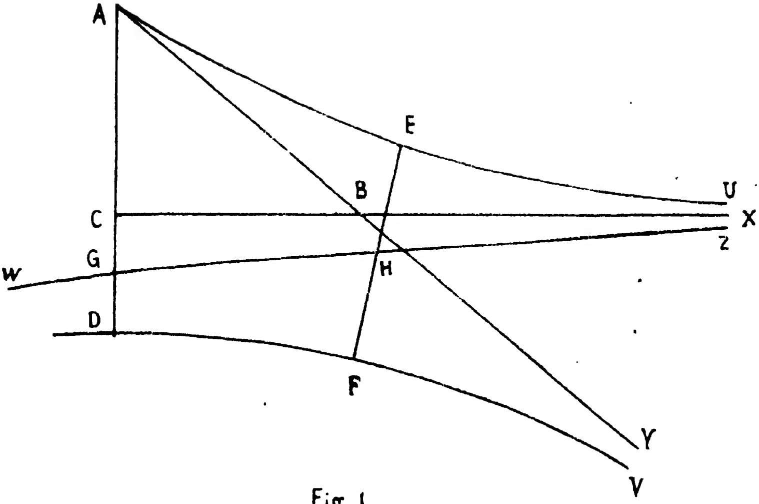


Fig. 1.

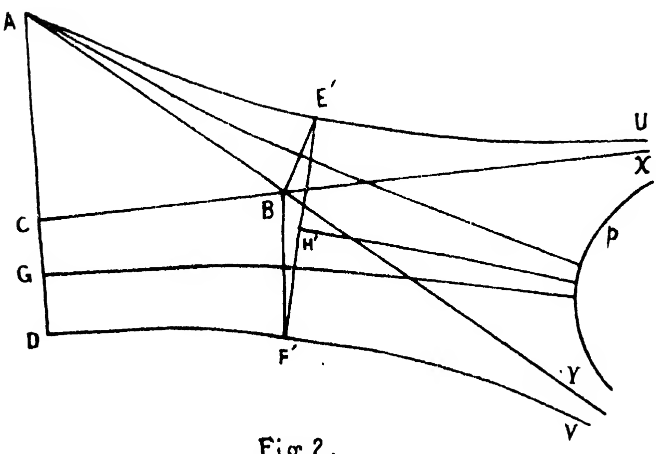


Fig 2.

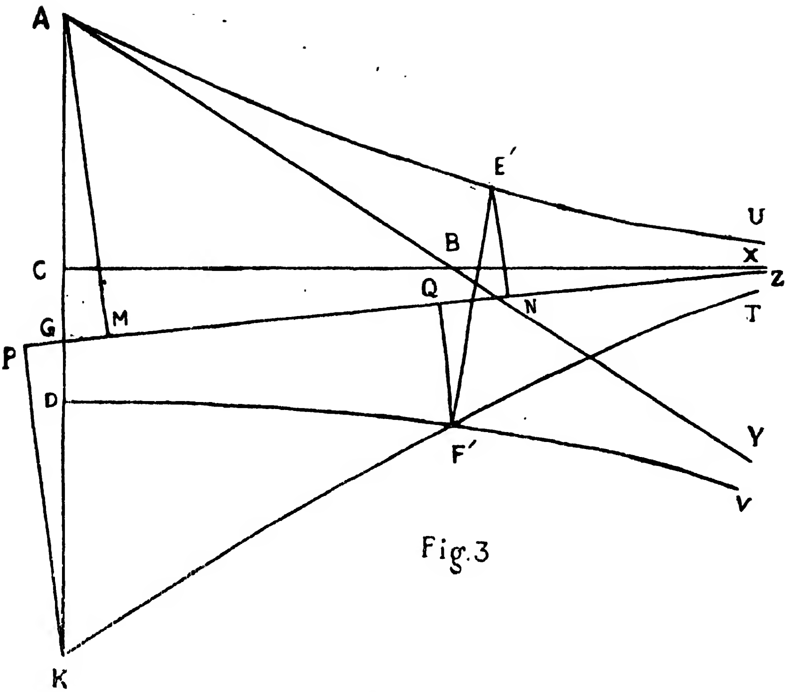


Fig. 3

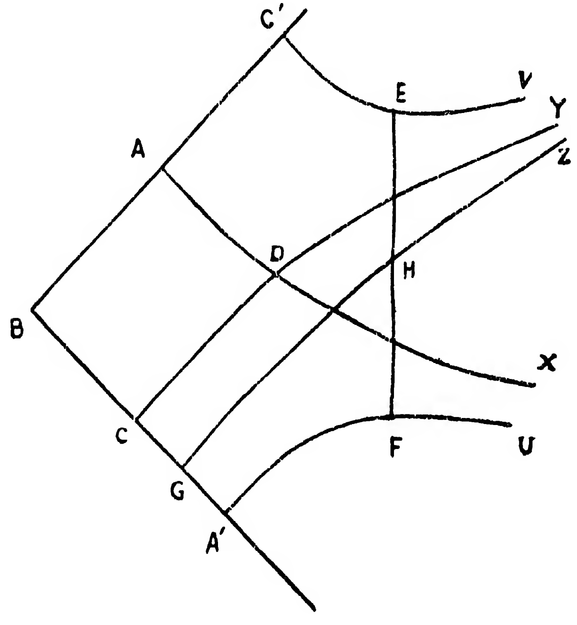


Fig. 4

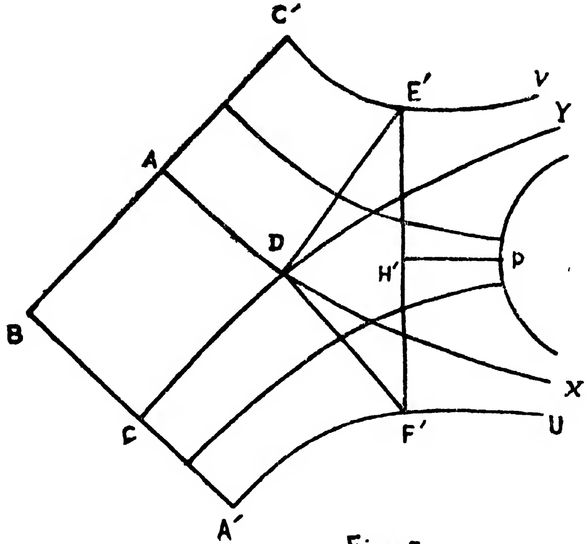


Fig. 5.

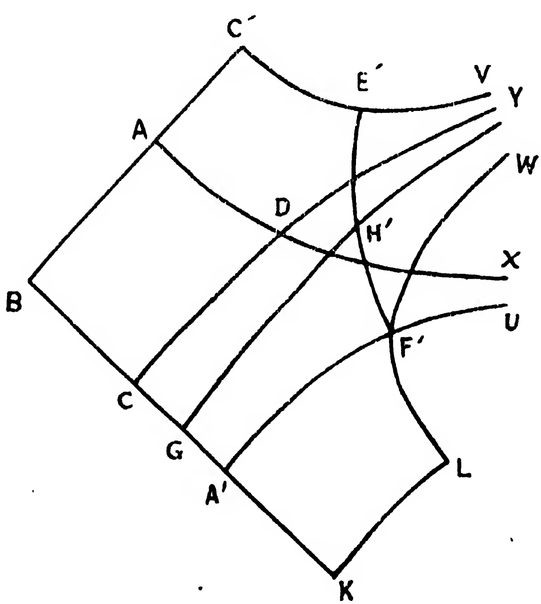


Fig. 6.

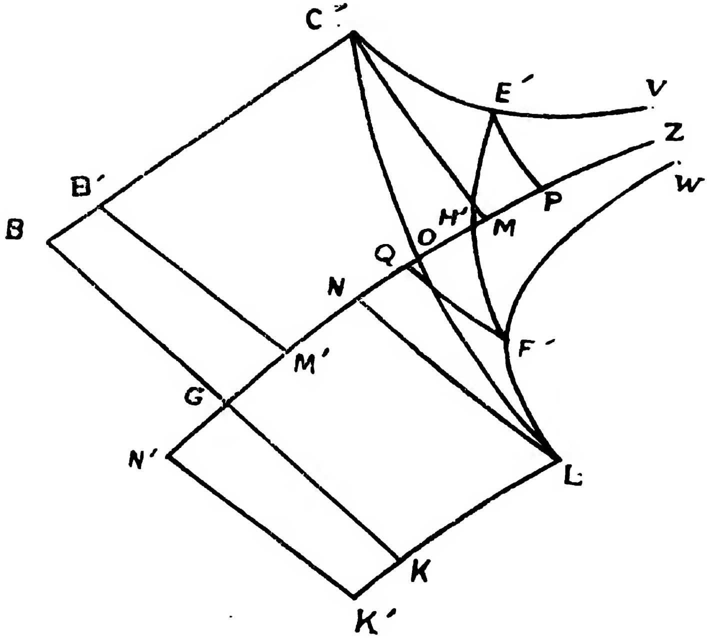


Fig. 7.

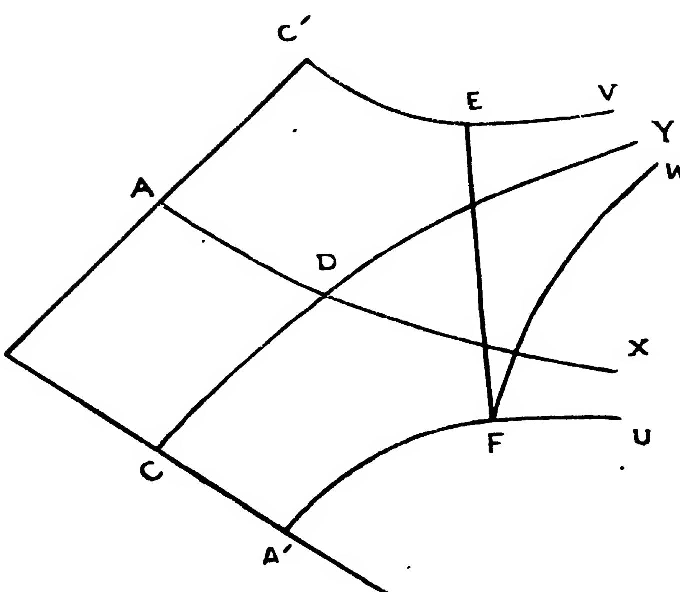


Fig. 8

NOTATION:—The angles of parallelism corresponding to lengths a, b, c, l, m are represented by the corresponding Greek letters α, β, γ λ, μ, and a', b', c', l', m' are lengths corresponding to angles of parallelism $\frac{\pi}{2}-\alpha$, $\frac{\pi}{2}-\beta$, $\frac{\pi}{2}-\gamma$, $\frac{\pi}{2}-\lambda$, $\frac{\pi}{2}-\mu$, and are called lengths complementary to a, b, c, l, m, n.

Theorem 1 gives the three-right-angled quadrilateral ADFE corresponding to the right-angled-triangle ABC. The angle DAE is β, being the angle of parallelism corresponding to distance AC which is b. The distance AD is l as it corresponds to the angle of parallelism CAB which is λ, and DF is equal to CB which is a. Also AE is equal to AB which is c.

The angle of parallelism corresponding to distance E′F′ is E′F″T, which is complementary to angle DF′K. But angle DF′K is equal to angle ABC which is μ. Therefore EF is m', being identical with E′F′. See Fig. 3.

COROLLARY 2, THEOREM 1.

Given a length l to construct the corresponding angle of parallelism λ. (Bolyai's classical construction.[1])

Take a length AD equal to l. Draw DF at right angles to AD and of any length. Draw FE at right angles to DF and draw AE perpendicular from A on FE. Thus DF and AE are obtained. Construct a right-angled-triangle with DF as base and AE as hypotenuse. The angle opposite to the base is the required angle λ, as is obvious from Theorem 1.

THEOREM 2.

ABCD is a three-right-angled quadrilateral, having right angles at A, B and C. Along BC and BA take lengths BA′ and BC′ complementary to BA and BC, respectively, so that A′U the parallel to AD through A′ makes a right angle with BA′ and C′V the parallel to CD through C′ makes a right angle with BC′. Let EF be the common perpendicular to A′U and C′V, meeting C′V at E and A′U at F. See Fig. 4.

Then AD is equal to C′E and CD is equal to A′F.

[1] For another geometrical proof of Bolyai's Parallel Construction, see Liebmann, Nichteuklidische Geometrie, 2nd Edition, p. 35, or, Carslaw, Non-Euclidean Geometry, p 73.

PROOF.

To avoid complicacy of constructions it would be convenient to divide the proof into several distinct parts.

(1)

Let G be the middle point of CA′ and H of EF.

Then GH is parallel to CD. See Fig. 4.

Proof similar to that of corresponding part of Theorem 1.

(2)

From C′V and A′U cut off C′E′ and A′F′ equal to AD and CD, respectively. Let H′ be middle point of E′F′.

Then angle C′E′F′ is equal to angle A′F′E′. See Fig. 5.

PROOF.

Consider the triangle E′F′D. Produce AD to X and CD to Y. The perpendicular bisectors of AC′ and A′C are also perpendicular bisectors of the sides F′D and E′D and are perpendicular to the common parallel p of AX and CY. Therefore the perpendicular to E′F′ at H′ is also porpendicular to p. Consequently from symmetry the angles C′E′F′ and A′F′E′ are equal.

(3)

GH′ is parallel to CD. See Fig 6.

PROOF.

Produce BA′ to K making A′K equal to BC so that G the middle point of CA′ is also the middle point of BK. Draw KL at right angles to BK and make KL equal to BA. Join F′L. The quadrilaterals ABCD and LKA′F′ are congruent and the angles ADC and LF′A′ are equal. Produce LF′ to W. Then LW is parallel to CY as AX is parallel to A′U.

Join C′L. Let O be the middle point of C′L. Through O draw OZ parallel to C′V. See Fig. 7.

Draw C′M, LN, E′P, F′Q perpendiculars to OZ. Let B′M′ and K′N′ be common perpendiculars between BC′ and OZ, and KL and OZ. Then C′M is equal to LN, as OC′ is equal to OL. Therefore angle MC′V equals angle NLW, and because C′E′ equals LF′ the figures MC′E′P and NLF′Q are congruent. Therefore E′P is equal to F′Q. Consequently OZ passes through H′, the middle point of E′F′.

Again MC'B and NLK are equal being supplements of the equal angles MC'V and NLW. Consequently MC'B'M' and NLK'N' are congruent. Therefore B'M' is equal to K'N'. The two figures BB'M'G and KK'N'G are congruent. Consequently BG is equal to GK, that is, OZ passes through G, the middle point of BK.

(4)

It follows from (1), (2), (3), that EF must coincide with E'F'.

This is established exactly as in the corresponding part of Theorem 1.

Thus Theorem 2 is completely proved.

Corollary 1, Theorem 2.

If we write the five elements a, b, c, λ, μ of a right-angled triangle in the order λ, μ, a, c, b there exists a rectangular pentagon whose sides in order are l, m, a', c, b'.

Suppose ABCD is the three-right-angled quadrilateral corresponding to a right-angled triangle with elements a, b, c, λ, μ so that AB, BC, CD, DA are equal to m', a, l, c and angle ADC is equal to β.

Construct the rectangular pentagon A'BC'EF as in Theorem 2. Then FA' A'B, BC', C'E are equal to l, m, a', c. The fifth side EF corresponds to angle of parallelism EFW which is complementary to angle UFW. But UFW is equal to XDY, that is β, so that EF is equal to b'. See Fig. 8.

Corollary 2, Theorem 2.

With each vertex of the rectangular pentagon as origin we can re-construct a three-right-angled pentagon and from this again a right-angled triangle. The sides of the rectangular pentagon may be written in order in five different ways;

$$l,\ m,\ a',\ c,\ b' \qquad (1)$$

$$m,\ a',\ c,\ b',\ l \qquad (2)$$

$$a',\ c,\ b',\ l,\ m \qquad (3)$$

$$c,\ b',\ l,\ m,\ a' \qquad (4)$$

$$b',\ l,\ m,\ a',\ c \qquad (5)$$

By identifying each of the sets (2), (3), (4), (5) with the set (1) we have five sets of possible values of a, b, c, λ, μ including the given set, *viz.*,

$$a, \quad b, \quad c, \quad \lambda; \quad \mu$$

$$c', \quad l', \quad b', \quad \mu, \quad \frac{\pi}{2}-a$$

$$b, \quad m', \quad l \quad \frac{\pi}{2}-a, \gamma$$

$$l', \quad a, \quad m, \quad \gamma, \quad \frac{\pi}{2}-\beta$$

$$m', \quad c', \quad a' \quad \frac{\pi}{2}-\beta, \lambda$$

We have thus the closed series of 5 associated right-angled triangles and the Engel-Napier Rules are shewn to possess a real geometrical basis in the rectangular pentagon.

Torsional Vibrations of a Circular Tube

BY

J. Ghosh.

1. The problem of the vibrations of cylinders has been discussed at great length by Rayleigh.[1] A particular solution in the case of the torsional vibrations of a solid cylinder has also been obtained. It is proposed in this paper to find the frequency equation in a more general case, *viz.*, when the solid is bounded by two co-axial cylinders and also when the thickness of the shell is small enough to be regarded as an infinitesimal of the first order.

2. Taking the axis of the cylinder as the axis of z and (r,θ,z) as the cylindrical coordinates of a point, the displacements of any point may be denoted by u_r, u_θ, u_z, which are usually assumed to be of the forms

$$\left.\begin{aligned} u_r &= \mathrm{U}e^{i(\gamma z+pt)} \\ u_\theta &= \mathrm{V}e^{i(\gamma z+pt)} \\ u_z &= \mathrm{W}e^{i(\gamma z+pt)} \end{aligned}\right\} \qquad \dots \quad (1)$$

where U, V, W are independent of z and t.

3. In our present problem, we have

$$u_r = u_z = 0 \text{ and } u_\theta = \mathrm{V}e^{i(\gamma z+pt)},$$

where V is a function of r only.

The equation of motion gives[2]

$$\frac{\partial^2 \mathrm{V}}{\partial r^2} + \frac{1}{r}\frac{\partial \mathrm{V}}{\partial r} - \frac{1}{r^2}\mathrm{V} + k^2\mathrm{V} = 0 \qquad \dots \quad (2)$$

[1] Theory of Sound, Vol. I, Chaps. VII, VIII.

[2] Love's Elasticity, Art. 200

where

$$k^2 = \frac{p^2 \rho}{\mu} - \gamma^2. \qquad \dots \quad (3)$$

The solution of (2) is evidently

$$V = AJ_1(kr) + BY_1(kr) \qquad \dots \quad (4)$$

where J_1 is the Bessel Function of the first kind and Y_1 the Bessel Function of the second kind, both of the first order.

The traction across any surface $r=r$ are given by

$$\widehat{rr} = \widehat{rz} = 0$$

and

$$\widehat{r\theta} = \mu \left[\frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r} \right]$$

Hence if the surface $r=r$ is free from tractions, we have

$$A \frac{\partial J_1(kr)}{\partial (kr)} + B \frac{\partial Y_1(kr)}{\partial (kr)} - \frac{1}{kr} \{AJ_1(kr) + BY_1(kr)\} = 0,$$

which, by means of identities

$$\frac{\partial J_n(z)}{\partial z} = J_{n-1}(z) - \frac{n}{z} J_n(z),$$

$$\frac{\partial Y_n(z)}{\partial z} = Y_{n-1}(z) - \frac{n}{z} Y_n(z),$$

reduces to

$$A[J_0(kr) - \frac{2}{kr} J_1(kr)] - B[Y_0(kr) - \frac{2}{kr} Y_1(lr)] = 0 \qquad \dots \quad (5)$$

Writing the conditions at $r=a$ and $r=b$ and elinimating A and B from these conditions, we get

$$\frac{kaJ_0(ka) - 2J_1(ka)}{kaY_0(ka) - 2Y_1(ka)} = \frac{kbJ_0(kb) - 2J_1(kb)}{kbY_0(kb) - 2Y_1(kb)} \qquad \dots \quad (6)$$

This equation determines k. We may get the value of γ from the conditions at the plane ends of the cylinder and the period p is then obtained by means of (3).

One solution of (6) is found to be $k=0$, and the corresponding solution for a cylinder clamped at $z=0$ and $z=l$, is

$$u_\theta=\left(C_n r+\frac{D_n}{r}\right)\sin\frac{n\pi z}{l}\cos\left(\frac{n\pi t}{l}\sqrt{\frac{\mu}{\rho}}+\alpha\right)$$

For a solid cylinder, we must put $D=0$.

4. We consider two particular cases.

(i) *One of the boundaries rigidly fixed.* If $r=b$ be a rigid boundary, we have from (4) and (5) the frequency equation

$$\frac{kaJ_0(ka)-2J_1(ka)}{kaY_0(ka)-2Y_1(ka)}=\frac{J_1(kb)}{Y_1(kb)}$$

(ii) *Thickness of the shell very small.* For a shell of radius a and thickness an infinitesimal of the first order, the frequency equation (6) may be replaced by the equation

$$\frac{\partial}{\partial a}\left[\frac{kaJ_0(ka)-2J_1(ka)}{kaY_0(ka)-2Y_1(ka)}\right]=0 \qquad \ldots \quad (7)$$

or $$[kaY_0(ka)-2Y_1(ka)][J_0(ka)+kaJ'_0(ka)+2J'_1(ka)]$$

$$-[kaJ_0(ka)-2J_1(ka)][Y_0(ka)+kaY'_0(ka)-2Y'_1(ka)]=0.$$

By means of the identities

$$\frac{\partial J_0(x)}{\partial x}=-J_1(x),\quad \frac{\partial Y_0(x)}{\partial x}=-Y_1(x)$$

$$\frac{\partial J_1(x)}{\partial x}=J_0(x)-\frac{1}{x}J_1(x)$$

$$\frac{\partial Y_1(x)}{\partial x}=Y_0(x)-\frac{1}{x}Y_1(x)$$

the above equation reduces to

$$k^2a^2[J_0(ka)Y_1(ka)-J_1(ka)Y_0(ka)]=0.$$

Since the value of the expression within brackets is of the form $\frac{c}{ka}$, where c is independent of ka, we get $k=0$ and this is the only solution of the equation (7). It is noteworthy that the relation between p and γ and hence the value of p itself is, in the case of a thin shell, independent of the radius of the shell.

On the Figures of Equilibrium of two rotating Masses of fluid for the Exponential Potential $\frac{e^{-kr}}{r}$.

Part II.

BY

Abanibhusan Datta.

1. In the first part of this paper, published in the Bulletin of the Calcutta Mathematical Society Vol. IX, No. 2, pp. 59-70., January, 1919, I studied the figures of equilibrium of two rotating masses of fluid for the exponential potential $\frac{e^{-kr}}{r}$, and intended to give later on a detailed numerical calculations of the results obtained therein and some diagrams, illustrating those results.

The present paper comprises the said numerical examples and some diagrams, showing the sections of the figures of equilibrium.

2. In part I, it has been shown that the equations of the two masses can be approximately written in the forms:

$$\frac{r}{a}=1+$$

$$\frac{\frac{20\pi}{k^3\sqrt{c}}\, i^{-\frac{1}{2}}\left(\mathrm{A}\cosh \mathrm{A}k-\frac{\sinh \mathrm{A}k}{k}\right)\mathrm{I}_{\frac{5}{2}}(ak)\,\mathrm{K}_{\frac{5}{2}}(kc)+\frac{1}{6}a^2\omega^2}{4\pi\left\{\frac{1}{k^2}\left(a\cosh ak-\frac{\sinh ak}{k}\right)\frac{e^{-ak}}{a}(ka+1)+af_{\frac{5}{2}}(ak)\,k_{\frac{5}{2}}(ak)\right.}\,p_2(\mu)$$

$$\frac{1}{12}a^2\omega^2 p_2^2(\mu)\cos 2\phi$$

$$4\pi\left\{\frac{1}{k^2}\left(a\cosh ak-\frac{\sinh ak}{k}\right)\frac{e^{-ak}}{a}(ka+1)+af_{\frac{5}{2}}(ak)\,k_{\frac{5}{2}}(ak)\right.$$

$$\frac{1}{k^2}\left(\mathrm{A}\cosh \mathrm{A}k-\frac{\sinh \mathrm{A}k}{k}\right)\frac{2}{\sqrt{ca}}\,i^{\frac{21}{2}}\left(\frac{7}{2}\right)\mathrm{I}_{\frac{7}{2}}(ak)\,k_{\frac{7}{2}}(kc)$$

$$\left(a\cosh ak-\frac{\sinh ak}{k}\right)\frac{e^{-ak}}{k^2 a}(ka+1)+af_{\frac{7}{2}}(ak)\,k_{\frac{7}{2}}(ak)$$

From symmetry, the equation of other mass is

$$\frac{R}{A}=1+$$

$$\frac{20\pi}{k^2\sqrt{c}}\, i \quad \left\{a\cosh ak-\frac{\sinh ak}{k}\right\} I_{\frac{5}{2}}(Ak)\, k_{\frac{5}{2}}(kc)+\frac{1}{6}A^2\omega^2 P_2(\mu)$$

$$4k\left\{\frac{1}{k^3}\left(A\cosh Ak-\frac{\sinh Ak}{k}\right)\frac{e}{A}^{-Ak}(ka+1)+Af_{\frac{5}{2}}(Ak)\, k_{\frac{5}{2}}(Ak)\right\}$$

$$\frac{1}{12}A^2\omega^2 P_2^2(\mu)\cos 2\phi$$

$$-\frac{\quad}{4\pi\left\{\frac{1}{k^2}\left(A\cosh Ak-\frac{\sinh Ak}{k}\right)\frac{e}{A}^{-Ak}(kA+1)+Af_{\frac{5}{2}}(Ak)\, k_{\frac{5}{2}}(Ak)\right\}}$$

$$+\frac{\frac{1}{k^2}\left(a\cosh ak-\frac{\sinh ak}{k}\right)\frac{2}{\sqrt{c}\,A}\, i^{\frac{2}{2}}\,\frac{7}{2}\, I_{\frac{7}{2}}(Ak)\, k_{\frac{7}{2}}(kc)}{\left(A\cosh Ak-\frac{\sinh Ak}{k}\right)\frac{e}{k^2A}^{-Ak}(kA+1)+Af_{\frac{7}{2}}(Ak)\, k_{\frac{7}{2}}(Ak)}$$

3. First let us consider the case when the masses are equal *i.e.* when $A=a$.

Let the distance between the centres be $c=2\cdot5a$. For these data, the figures of the two masses are similar in shapes.

When $ka=\cdot5$, each is given by an equation of the form :

$$\left.\begin{aligned}\frac{r}{a}&=1+\cdot19706\, p_2(\mu)-\cdot021\, p_2^2(\mu)\cos 2\phi+\cdot01164\, p_3(\mu)\\ \frac{R}{A}&=1+19706\, P_2(\mu)-\cdot021\, P_2^2(\mu)\cos 2\phi+\cdot01164\, P_3(\mu)\end{aligned}\right\}\quad(1)$$

Again, when $A=a$, $c=2\cdot5a$, and $ka=1$ the figures are given by

$$\left.\begin{aligned}\frac{r}{a}&=1+\cdot16287\, p_2(\mu)-\cdot0144\, p_2^2(\mu)\cos 2\phi+\cdot02135\, p_3(\mu)\\ \frac{R}{A}&=1+\cdot16287\, P_2(\mu)-\cdot0144\, P_2^2(\mu)\cos 2\phi+\cdot02135\, P_3(\mu)\end{aligned}\right\}\quad(2)$$

Also, when $A=a$, $c=2\cdot5a$, and $ka=10$, the figures are given by

$$\left.\begin{aligned}\frac{r}{a}&=1+\cdot0002845\, p_2(\mu)-\cdot0000119\, p_2^2(\mu)\cos 2\phi+\cdot000014823\, p_3(\mu)\\ \frac{R}{A}&=1+\cdot0002845\, P_2(\mu)-\cdot0000119\, P_2^2(\mu)\cos 2\phi+\cdot000014823\, P_3(\mu)\end{aligned}\right\}\quad(3)$$

On plotting (1), [*cf.* Table I, and Fig. I], it is found that the figures are almost similar in shapes to those given by Darwin with similar data. If however, ka is increased, the figures tend to become more spherical in shapes as will appear from (1), (2) and (3). When ka is fairly large as in (3), the terms involving the different harmonics are almost of negligible order of smallness.

Similar peculiarities are also noticed when the masses are unequel. When $A=3a$, and $\frac{c}{a}=5{\cdot}3$ and $ka={\cdot}2$, it is found that the figures are given by

$$\left.\begin{aligned}\frac{r}{a}&=1+2881\,p_2(\mu)-{\cdot}0234\,p_2^2(\mu)\cos 2\phi-1{\cdot}079\,p_3(\mu)\\ \frac{R}{A}&=1+{\cdot}06558\,P_2(\mu)-0029\,P_2^2(\mu)\cos 2\phi-{\cdot}000272\,P_3(\mu)\end{aligned}\right\}\quad(4)$$

The figures, as given by (4) have been drawn [*cf.* Table II, Fig. II and III]. The curves, however, do not present any marked difference from those obtained by Darwin for Newton's law with similar numerical data

TABLE I. (For Figure I).

θ	$\cdot 197\,p_2(\mu)$	$-\cdot 021\,p_2^2(\mu)$	$\cdot 01164\,p_3(\mu)$	r/a
0°	·197	0	·01164	1·209
15°	·1772	—·0042	·0094	1·1824
30°	·122	—·015	·0038	1·1108
45°	·049	—·0315	—·002	1·0155
60°	—·0246	—·0473	—·005	·941
75°	—·079	—·0485	—·004	·8685
90°	—·0985	—·063	0	·8385
105°	—·079	—·0485	·004	·9125
120°	—·0246	—·0473	·005	·951
135°	·049	—·0315	·002	1·0195
150°	·122	—·015	—·0038	1·103
165°	·1772	—·0042	—·0094	1·1636
180°	·197	—0	—·01164	1·185

TABLE II. (For Figures II and III).

θ	$\cdot 2881\, p_2$	$-\cdot 02348\, p_2^2$	$-\cdot 1079\, p_3$	r/a
0°	·2881	0	—·1079	1·18
15°	·2591	—·0047	—·0868	1·1676
30°	·18	—·0176	—·035	1·1274
45°	·072	—·035	·019	1·056
60°	—·036	—·0528	·047	·9582
75°	—·1151	—·0554	+·037	·8667
90°	—·14405	—·07	0	·786
105°	—·1151	—·0554	—·007	·8
120°	—·036	—·0528	—·047	·8642
135°	·072	—·035	—·019	1·018
150°	·18	—·0176	+·035	1·1974
165°	·2591	—·0047	·0868	1·3412
180°	·2881	0	·1079	1·396
...	$\cdot 06558 P_2$	$-\cdot 0029 P_2^2$	$\cdot 000272 P_3$	R/A
0°	·06558	0	·000272	1·0658
15°	·05899	—·000058	·00022	1·059
30°	·0401	—·002175	·000088	1·038
45°	·0164	—·00435	—·00005	1·012
60°	—·0082	—·0065	—·00012	·9852
75°	—·0262	—·0081	—·00094	·965
90°	—·0328	—·0087	0	·9585
105°	—·0262	—·0081	·00094	·967
120°	—·0082	—·0065	·00012	1·0146
135°	·0164	—·00435	·00005	1·012
150°	·0401	—·002175	—000088	1·038
165°	·05899	—·00058	—00022	1·058
180°	·06558	...	—000272	1·0653

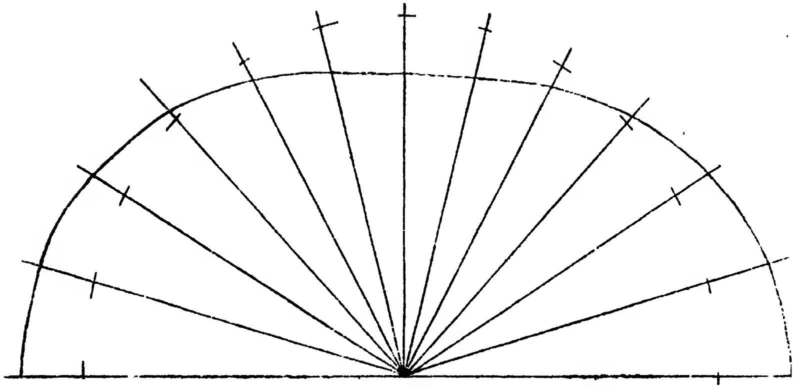


Fig. I

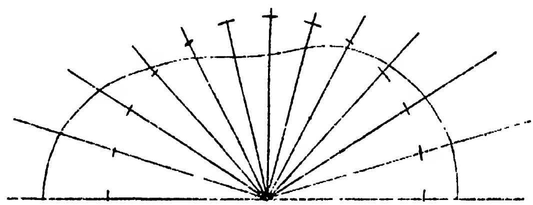


Fig. II

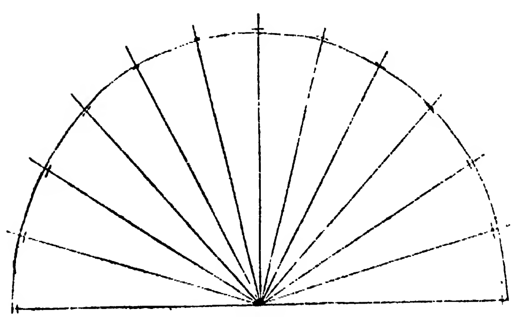


Fig. III

BOTANY

Commentationes Algologicae

II

Algæ Epiphyticæ Epiphloiæ Indicæ

OR

Indian Bark Algæ

BY

P. Brühl and K. P. Biswas.

Myxophyceæ.

Chroococcus turgidus (Kuetz.) Nægeli.

For the description of this species see "Algæ of Bengal Filter-beds" in the Science Journal of the Calcutta University, Vol. IV, page 2.

See also plate 1, fig. 1 of the present contribution.

The contents of the specimens found growing on the bark of trees are granular and of a more or less distinctly brownish tint.

Chroococcus minutus (Kuetz.) Naegeli, forma typica.

Cellulis globosis vel ante divisionem ellipsoideis, saepe mutua pressione plus minusve angulosis, plerumque geminatim vel quaternatim approximatis, sine tegumento 3-5μ, cum tegumento 3-6μ diametro; familiis e cellulis 2, 4, 8, 16, 32, sæpissime 8 compositis; familiis 8-cellularibus cum tegumento communi suborbiculari hyalino 9-13μ longis, 6-12μ latis; contentu cellularum semper granuloso, aerugineo vel pallide brunneo-viridi.

Habitat ad corticem *Terminaliae Catappae.*

Cells globose or ellipsoidal, by mutual pressure often more or less angular, usually in twos, fours or eights closely approximate, but also

in groups of 16 or even 32, usually eight united into a family 9-13μ long, 6-12μ wide, surrounded by a common hyaline integument; the single cells without tegument 3-5μ, with tegument 3-6μ in diameter; contents of cells always granular, pale verdigris-green or pale brownish green.

Plate I, fig. 2.
De Toni, Myxophyceæ, p. 14.
Rabenhorst, Flora Europæa Alg., II, p. 30.
Hansgirg, Prodr. II, p. 162.

APHANOCAPSA BRUNNEA, NAEGELI.

Forming a thin, gelatinous, brown layer of indefinite outline closely adhering to the bark. Cells spherical or ellipsoidal, before division roundish-oblong, 4-6μ in diameter, often 4-5μ across and 5-6μ in length, either singly or in pairs; teguments thin, hyaline, diffluent; cell-contents pale brown or brown with a greenish tint, granular.

Found growing on *Terminalia Catappa* and *Enterolobium Saman* during the rains, mixed with other bark algae.

See Plate I, fig. 3.
De Toni, Myxophyceae, p. 71.

MICROCYSTIS MARGINATA, Kuetzing.

Colonia globoso-lenticulari aut omnino sphaerica (sæpe coloniis pluribus contiguis sed vix confluentibus), pallide cæruleo-viridi, 14-100 (—180)μ diametro, margine diaphano interdumque modice interrupteque lamelloso, 1·5-2μ lato; cellulis 3-4μ diametro, dense aggregatis in familias multicellulares; contentu granuloso, pallide cœruleo-viridi.

Arcte adhærens cortici *Terminaliae Catappae, Mangiferae indicae, Enterolobii Saman, Swieteniae Mahagoni.*

Lecta 22-8-22.

Colonies lenticular-globose or quite spherical, often a number of them close together, but rarely confluent, pale bluish green, usually between 14 and 100μ in diameter, but sometimes reaching 180μ, the margin of the colony being translucent, sometimes sparsely and interruptedly lamellose, 1·5-2μ wide; the cells 3-4μ in diameter, densely crowded into a many-celled family; contents granular, pale bluish-green.

Closely adhering to the bark of various trees.

Plate I, fig. 4.

De Toni, Myxophyceae, p. 91.

Kuetzing, Tab. Phyc. I, pl. 8.

Cooke, Brit. Freshw. Algæ, pl. 86.

OSCILLATORIA KUETZINGIANA, Naegeli.

Strato laevi, continuo, in siccitate papyraceo, nitido, circiter 0·5 mm. crasso, viridi-nigrescente, aqua madefacto caeruleo-viridi, gelatinoso; trichomatibus inter se parallelis, cortici arcte adhaerentibus, rectis, apice sensim modiceque attenuatis neque capitatis, ad genicula constrictis, 1·5-2μ crassis, aut prorsus progredientibus aut parte superiori oscillantibus; articulis 2·5-3μ longis; contentu caeruleo-viridi, minute granuloso, cum granulis crassioribus uno vel duobus centralibus et duobus dissepimento utroque approximatis; dissepimentis conspicuis.

Crescit ad corticem *Tectonae grandis* in suburbe Baliganj dicta Calcuttensi.

Lecta 12-7-1922.

Forming a smooth, continuous covering on the bark of *Tectona grandis*, the covering being, when dry, papery, shining, about half a millimeter thick, greenish-black, when moistened with water bluish-green, slimy; the filaments parallel to each other, closely adhering to the bark, straight, somewhat attenuated towards the apex, not capitate, constricted at the joints, 1·5-2μ thick, moving straight onwards or oscillating to and fro with their upper part; cells 2·5-3μ long; contents bluish-green, finely granular and with one or two thicker granules near the centre and two similar granules near each dissepiment; dissepiments conspicuous.

Found growing on the bark of *Tectona grandis* in Baliganj.

Plate I, fig. 5, a, b.

De Toni, Myxophyceae, p. 170.

Occurs in various parts of Europe and in Australia.

OSCILLATORIA ACULA, Brühl et Biswas, *species nova.*

Trichomatibus solitariis vel in fasciculos paralleliter aggregatis, vix fragilibus, ad genicula haud constrictis, 4-6μ crassis, 70-400μ longis, plerumque strictis, apice acutatis vel acuminatis, subobtusis, haud capitatis neque calyptratis, saepissime plus minusve uncinatis,

rarissime omnino rectis; articulis 3-4μ longis; contentu caeruleo-viridi, tenuiter granuloso, interdum cum granulis paullo maioribus superficiei approximatis intermixtis.

Occurrit intermixta *Lyngbiae arboricolae* et *Lyngbyae connectenti* ad corticem *Enterolobii Saman*, *Terminaliae Catappae* et specierum *Fici.*

Affinis *Oscillatoriae animalis*, sed minus fragilis, haud agglomerrata in stratum aeruginosum, sed semper aut singula aut plures parallele in fascicula consociata, trichoma apice rectum vel subabrupte uncinatum.

Trichomata either solitary or a number of them parallel to each other aggregated into bundles of moderate size, hardly brittle, not constricted at the joints, 4-6μ thick, 70-400μ long, usually quite straight, narrowed or acuminate towards the subobtuse, non-capitate, non-calyptrate apex, which may be straight, but is more often rather abruptly bent aside; cellules 3—4μ long; their contents bluish-green, finely granular, sometimes with some larger granules close to the surface.

On the bark of various trees accompanying *Lyngbya arboricola* and *Lyngbya connectens.*

This species is near *Oscillatoria animalis*, but the trichomes are less fragile and never, as far as known, form an expanded continuous stratum, but occur as single filaments scattered among other algæ or aggregated side by side into small fascicles; also, the trichomes, although sometimes straight throughout, are usually rather abruptly bent sideways near one or both ends.

Plate I, fig. 6, a, b.

LYNGBYA CONNECTENS, Brühl et Biswas, *sp. nova.*

Stratum extensum, continuum, corticem arborum vestiens, tenue, circiter 1 mm crassitudine, in siccitate saepius nitidum, nigro-aeruginosum; trichomatibus rectis vel subrectis, parallelis, in iuventute nudis, vagina hyalina in vetustate fuscescente vel plane fusca vestitis, siccitate fragilibus, ad genicula haud constrictis, 12—17μ crassis, apice vix attenuatis et saepissime rotundatis, interdum paullo incrassatis; vagina firma, 1.5—2μ crassa, homogenea vel rarius e lamellis 2—3 composita; articulis 2—2.5μ longis, longitudine circiter sexta pars latitudinis; dissepimentis granulatis; contentu granuloso, aerugineo.

Habitat ad corticem *Fici infectoriae* et speciei Fici indeterminatae in suburbibus Calcuttensibus et Lucknowensibus.

Lecta Julio 1922 et Ianuario 1923.

Forming an extensive coating on the bark of *Ficus infectoria* and an undetermined species of Ficus, the stratum being about one millimeter thick, when dry shining and dark verdigris-green. Filaments straight or nearly so, lying parallel to each other, the trichomes often creeping out of their entire sheath, the alga then being scarcely distinguishable from *Oscillatoria limosa*, Agardh ; when older the trichomes cover themselves with a sheath which is at first delicate and colourless, but which with age becomes firm and assumes a brown tint ; trichomes not constricted at the joints, slightly thickened at the apex ; cellules 2—2·5μ long, the length being about one-sixth of the diameter ; dissepiments granulated ; contents granular, verdigris-green.

Plate II, fig. 8, a-e.

LYNGBYA AESTUARII (Marteus) Liebman, *var.* ARBUSTIVA Brühl et Biswas, *var. nova.*

Stratum expansum, arborum corticem vestiens, tomentosum, 1—10 mm. crassitudine, in siccitato colore fusco vel badio, pluvio humectatum caeruleo-viride ; filamentis elongatis, flexilibus basi plus minusve prostratis, sinuosis, plus minusve dense intertextis ; vagina initio tenui, brevi, achroa, aetate provecta fusca vel badia, vetustate pallidiori aut plane achroa, 20—28μ externo diametro, 2—5μ crassitudine, extus plus minusve rugosa, interdum annulis tenuibus 15—18μ distantibus cincta, lamellosa, lamellis 2—10 ; trichomatibus 17—18μ diametro, ad dissepimenta haud vel vix constrictis, apice rotundatis neque conicis neque attenuatis capitatisve ; articulis 4—6, raro 9μ longis ; contentu pallide smaragdino, dense granuloso, granulis partim minutissimis, partim paulo crassioribus, translucentibus, saepe prope dissepimenta aggregatis.

Habitat ad corticem *Swieteniae Mahagoni, Tectonae grandis, Enterolobii Saman* et *Fici bengalensis.*

Forming an extensive coating on the bark of various trees, the coating being felty, from 1—10 mm. thick, when dry of brown tints, when wetted by rain bluish-green ; filaments long, flexible, at the base more or less prostrate, sinuous, more or less densely interwoven ; sheath at first thin and colourless, later on greyish or reddish brown,

20—28μ in external diameter, 2—5μ thick, rough on the outer surface, lamellose, the number of lamellae being two to ten; trichomes 17—18μ in diameter, scarcely or not at all constricted at the joints; articulations usually 4—6, rarely 9μ long; contents pale emerald-green, densely granular, granules partly minute, partly somewhat larger, often more crowded near the dissepiments.

The alga grows vigorously when placed in a dilute solution of sodium chloride. According to De Toni it grows in brackish water, in peat swamps and near hot springs and is cosmopolitan.

The following is translated from Maurice Gomont's "Monographie des Oscillariés" (Annales des Sciences Naturelles: Botanique, Vol. XVI, 1892, pp. 30, 31).

"*Lyngbya aestuarii* adapts itself to the most diverse media as regards chemical composition. It is found not only at the mouth of rivers, as its name indicates, but also, and in great abundance, in the saturated waters of salt swamps. It has been gathered in hot springs and also in fresh water. As it occurs in similar cases, its synonymy is complicated. But if we except variations in diameter, the various forms are merely passing modifications due to age and the conditions in which it grows. For instance, if the plant develops at a certain depth, the sheaths remain hyaline and allow the colour of the trichome to be clearly seen. The sheaths become red-brown and communicate this tint to the plant-mass, when the latter floats on the surface. The names *Lyngbya aeruginosa* and *L. ferruginea* correspond to these different states. On moist soil the filaments lengthen more freely than when immersed in water and interlace in thousand different ways. We have then *L. pannosa. L. curvata*, *L. congesta* and others. Sometimes the filaments become erect and form a mesh work. We have then *Symploca crispata*, *Symphyosiphon leucocephalus*. As regards thickness, the extreme forms are connected by an uninterrupted series of connecting links, so that divisions which one would feel inclined to establish on the basis of that character, would be absolutely artificial."

We may add that the form described by us has been observed at various places in the southern suburbs of Calcutta, growing on the bark of various tall trees. During heavy rains the bark of these trees is closely covered on all sides with a bluish-green lining of the algæ,

which lining becomes dark reddish-brown after the cessation of the rains.

Plate II, fig. 9, a-c.
See De Toni, Myxophyceæ, p. 262.
Cooke, Brit. Freshw. Algæ. pl. 101.
Wolle, Freshw. Alg. U.S., pl. 200.

LYNGBYA ARBORICOLA, Brühl et Biswas, *sp. nova.*

Stratum corticem arborum vestiens, plus minusve extensum, humidum aerogineo-viride, in siccitate tenue, fulvum, densissime tomentosum et velutinum, minus 1 mm. crassitudine, in atmosphaera humida cultum sublanosum, ad 6 mm. crassum, pelli simile, e filamentis vivis atque vaginis vacuis achrois aut fuscis constans; filamentis subrectis vel modice curvatis et dense intertextis, elongatis, cum vagina 18—22μ crassis; trichomatibus ad genicula leviter sed manifeste constrictis, aerugineis, ad extremitatem rotundatis; vagina firma, 1·5—2μ crassa, in statu iuvenili achroa, laevi, homogenea, aetate provecta saepius fusca et haud raro e stratis duobus composita, saepe facie externa minute transverse rugulosa; articulis 16—21μ crassis, saepissime 5—6μ, rarius 6—10μ longis; dissepimentis saepissime manifestis, rarius obscuris vel plane evanidis; contentu aeruginoso, dense granuloso; hormogoniis ex articulis 2 ad 20 compositis.

Habitat ad corticem *Enterolobii Saman, Tamarindi indicae, Fici religiosae, Swietoniae Mahagoni, Terminaliae Catappae, Tectonae grandis, Mangiferae indicae,* in urbe atque suburbibus Calcuttensibus, in horto botanico Sibpurensi, in ditione Mymensing (legit S. P. Nag) et Lucknow.

Stratum coating the bark of various trees, when moist verdigris-green, when dry more or less reddish-brown, in both states velvety to the feel, forming a continuous layer consisting of living filaments and empty sheaths; sheaths colourless at first or, after having been empty for some time, mostly reddish-brown; filaments nearly straight or moderately flexuous, long, with the sheath 18—22μ in diameter; the trichomes shallowly but distinctly constricted at the joints, verdigris-green, rounded at the extremity; sheath firm, 1·5—2μ thick, homogeneous or slightly stratified, often slightly transversely wrinkled, articulations usually 5—6μ, sometimes 6—10μ long; contents verdigris-green, densely granular.

Found on various trees.

It may be held to be related to *Lyngbya aestuarii*, but differs from the latter by the thallus being quite continuous, usually bluish-green and always velvety to the feel (not rough to the feel and more or less interrupted and having a reddish-brown tint), the trichomes being distinctly constricted at the joints (not without constrictions), the sheath consisting at most of three lamellae (not of 3—10 very distinct layers) and being 1·5—2μ (not 2—5μ) thick.

Plate III, fig. 10, a-e.

LYNGBYA DENDROBIA, Brühl et Biswas, *sp. nova.* **Forma typica.**

Stratum plus minusve expansum, compactum, tenue, minute densissimeque tomentosum, cortici arborum arcte adhaerens, tempore pluviarum colore cinereo vel fuscescente, temporibus siccis colore badio vel pullo; filis elongatis, flexilibus, modice flexuosis, arcte intertextis, cum vagina 10—11μ crassis; vagina initio atque aetate provecta tenui, 1—1·5μ crassa, extus laevi, hyalina, aetate provecta vulgo haud fuscescente neque stratosa, raro fuscescente, modice lamellosa; trichomatibus ad genicula haud vel obscure constrictis, apice rotundatis vel levissime attenuatis, 9—10μ crassis, articulis 4—6μ longis (4·5 × 10, 6 × 10, 4 × 10, 5 × 10μ), diametro 1·7—1·5 longitudinis; contentu fulvo vel fusco, uniformiter denseque granuloso; dissepimentis conspicuis, haud notatis serie granulorum; membrana cellulae apicalis haud incrassata.

Habitat ad corticem *Tectonae grandis*, *Enterolobii Saman*, *Swieteniae Mahagoni*, *Mangiferae indicae*, *Fici religiosae*, *Fici bengalensis*, *Oriodoxae regiae*, in suburbibus Calcuttensibus, et ad corticem *Meliae Azidarachtae*, *Tamarindi indicae* et *Fici religiosae* in Lucknow et Burdwan.

Forma LURIDA. Uti in forma typica, articulis 10—12μ diametro, 6—8μ longis; vagina circiter 0·6μ crassa, hyalina; contentu cellularum livido, olivaceo-cinerascente, interdum flavescente vel purpurascente, dense conspicueque granuloso, granulis crassioribus quam in forma typica. Stratum filamentis vivis discoloribus atque vaginis vacuis achrois arcte intermixtis, colorem cinereum vel canum praebens. In iisdem locibus pluviarum tempore.

Stratum more or less expanded, compact, thin, minutely and densely tomentose; filaments long and flexible, closely interwoven, with sheath 10—11μ thick: sheath usually thin, 1—1·5μ thick,

smooth, hyaline, usually colourless, more rarely when old brownish and very moderately stratified : trichomes scarcely or not at all constricted at the joints ; cellules 4—3μ long, 1.7—2.5 times as broad ; contents of various shades of brown, uniformly and densely granular ; dissepiments conspicuous, not marked by granules.

During the rains the colour of the stratum is greyish-brown or of various shades of grey, in the dry seasons the colour is reddish brown. The forma *lurida* is found during the raius.

Plate III, fig. 11, a-d.

LYNGBYA CORTICICOLA, Brühl et Biswas, *sp. nova.*

Strato tenui, arborum corticem vestiente, dense tomentoso, fusco vel fulvo ; filamentis subfragilibus, modice flexuosis, plus minusve intricatis, cum vagina 12—16μ crassis ; vagina initio hyalina, mox fuscea, 2μ crassa, haud vel vix lamellosa, superficie irregulariter aspera ; trichomatibus 8—12μ crassis, ad dissepimenta levissime constrictis, longitudine dimidia vel tertia pars latitudinis, apice rotundatis ; contentu aerugineo, granuloso ; dissepimentis granulis haud notatis.

A *Lyngbya aestuarii* differt filamentis subfragilibus neque flexilibus, vaginis vix vel haud lamellosis, trichomatibus ad dissepimenta leviter constrictis, articulis diametro 2—3 plo (nec 3-6-plo) brevioribus, vaginis asperis (nec rugulosis).

Vestiens corticem *Terminaliae Catappae* in horto botanico Sibpurensi.

Lecta 22-8-1922.

Forming a thin, densely tomentose, dark or yellowish-brown layer on the bark of *Terminalia Catappa* ; filaments somewhat fragile, moderately flexuous, more or less closely intricate, including the sheath 12—26μ thick; vagina at first hyaline, but soon becoming brown, 2μ thick, scarcely or not at all lamellose, with a rough, uneven, but not wrinkled surface; trichomata 8—12μ in diameter, slightly constricted at the joints, about ½ to ⅓ as long as wide, rounded at the apex; contents of the cells verdigris-green, granular ; dissepiments not marked by prominent granules.

Plate IV, fig. 13, a-c.

LYNGBYA AERUGINEO-CAERULEA, (Kuetzing) Gomont.

Stratum obscure aerugineum; filamentis saepe singulis, aliis algis intermixtis, flexuosis; vaginis tenuissimis, hyalinis; trichomatibus 3-6. μ crassis, pallide aerugineo-caeruleis, ad genicula haud constrictis; articulis 2-4 μ longis, saepius diametro brevioribus, rarius diametro aequilongis; contentu uniformiter et grosse granuloso, granulis saepe dissepimenta obducentibus; cellula apicali rotundata vel leviter conica, membrana leviter incrassata.

Occurrit ad corticem *Enterolobii Saman*, *Terminaliae Catappae* et *Swieteniae Mahagoni* in Calcutta et in horto botanico Sibpurensi. Lecta Julio 1922.

Forming a thin stratum on the bark of various trees or occurring in similar positions singly among other bark algae. Filaments flexuous. Trichomata 3-6 μ thick, pale bluish-green, not constricted at the joints; cellules 2-4 μ long, usually shorter, never longer than wide; contents uniformly and rather coarsely granular, the granules often nearly obscuring the dissepiments; apical cell rounded or slightly conical at its outer end, very slightly thickened.

De Toni, Myxophyceae, p. 281.
G. S. West, Algae, vol. I, p. 42, fig. 28, B.C.
Plate III, fig. 12, a-c.

LYNGBYA SUBTILIS W. West.

var. TYPICA.

Filamentis solitariis et sparsis vel in societates laxas congregatis, tenuibus, elongatis, flexuosis, ad genicula haud constrictis, strato Tolypothricis byssoideae ad corticem *Terminaliae Catappae* intermixtis, 1·5-2 μ crassis; vagina tenuissima, minus 0·5 μ crassa, hyalina, achroa; articulis 3-4 μ longis, diametro circiter duplo longioribus; contentu pallide aerugineo, evidenter homogeneo; dissepimentis aegre conspicuis; hormogoniis e tribus vel pluribus cellulis efformatis.

Lecta. 22.7.22. In cultura viva in Decembro 1922.

Filaments solitary among Tolypothrix byssoidea on the bark of Terminalia Catappa, also forming lax associations among themselves, thin, long and flexuous; sheath exceedingly thin, hyaline and colourless; the cellules 3-4 μ long, about twice as long as wide; contents

pale verdigris-green, apparently homogeneous; dissepiments difficult to distinguish.

De Toni, Myxophyceae, p. 285.
Plate I, fig. 7c.

LYNGBYA SUBTILIS W. West, *var.* GRANULOSA, Brühl et Biswas.

Differt a varietate typica contentu minutissime sed manifeste granuloso. Filamenta saepissime solitaria, diametro 2-3 μ; dissepimenta manifesta; cellulae saepe diametro duplo longiores; vagina achroa, aliis algis (Tolypothrici byssoideae, Lyngbyae dendrobiae et Lyngbyae arboricolae) intermixta ad corticem *Terminoliae Catappae, Enterolobii Saman, Swietenia Mahagoni* et *Fici infectoriae,* in suburbibus Calcuttensibus et in horto botanico Sibpurensi.

Lecta Julio 1922.

The present variety is distinguished from the typical variety by the contents of the trichomes, as seen under higher powers of the microscope, being clearly granular.

Plate I, fig. 7, a, b.

LYNGBYA PALMARUM, Brühl et Biswas, (=SCYTONEMA PALMARUM, Martens.)

Stratum extensum, dense minuteque tomentosum vel velutinum, saturate aerugineum, in siccitate pallide viridi—caeruleum vel pallide flavescens, faciem exteriorem vaginarum vel cicatrices foliorum delapsorum palmarum vestiens; filamentis substrato applicatis et ascendentibus, subrectis vel modiee curvatis sinuatisve, laxiuscule vel arctissime intertextis, cum vagina 6-10 μ crassis; vagina initio atque aetate provecta laevi, achroa, hyalina, aetate provecta saepe discolori neque fuscescente, 0·6-1 μ crassa; trichomatibus ad genicula haud constrictis, apice rotundatis vel vix attenuatis; articulis 4-5-8 μ longis (l : b : : 8 : 8, 6 : 6, 8 : 6 μ); contentn aerugineo, aetate provecta interdum fulvescente, dense granuloso; dissepimentis paullum conspicuis, saepe granulis fere occultatis, sed serie granulorum haud notatis; membrana cellulae apicalis haud incrassata.

Occurrit in horto botanico Sibpurensi et in palmetis burmanicis.

Lecta Julio 1922.

Forming a continuous, minutely and densely felty or softly velvety coating on the lower surface of the leaf-sheaths or the

cicatrices of fallen leaves of Phœnix sylvestris and certain other palms. Filaments nearly straight or flexuous, usually densely interlaced.

Kurz' collections, No. 1789 :

" *On Phœnix sylvestris, in the cicatrices of the fallen leaves, wherein water accumulates. Calcutta 7, 67.*"

Plate IV, fig. 14.

LINGBYA CALCIFERA, Brühl et Biswas, *sp. nova.*

Stratis initio parvis, rotundatis, 1-2 mm. diametro, demum in stratum extensum 0.5—2 mm crassum, lichenoideum, plus minusve pustulatum, arborum corticem vel calcem tectorium murorum vestientem, in siccitate colore cæsio, aqua conspersum plus minusve nigrescentem consociatis; filamentis calce incrustatis, sine crusta calcaria circiter 8-10μ crassis, crusta calcaria crassitudine valde inaequali, 2-9μ crassa, alba vel cinerea, superficie aspera; filamentis plus minusve flexuosis, apice rotundatis; vagina circiter 2μ crassa, saepe lamellosa e lamellis 2-5 constructa, colore fusco; cellulis circiter 4μ diametro, 6-10μ longis; contentu viridi-cæruleo, fere æquabiliter granuloso.

Habitat ad corticem *Aegles Marmelos, Terminaliæ Catappæ, Pterocarpi sp.*, atque ad muros umbrosos urbis Calcuttæ.

Filaments at first aggregated into small roundish clusters, 1-2 mm. in diameter, which finally fuse into an extensive, lichenlike, 0.5—2 mm. thick, pustulate stratum, ashy-grey when dry, rather black when wetted; filaments incrusted by calcium carbonate, without the crust 8-10μ thick more or less flexuous, rounded at the apex; vagina about 2μ thick, usually lamellose and then consisting of 2-5 lamellæ, brown; calcareous crust very uneven in thickness, 2-9μ thick, rough on the surface; cells about 4μ in diameter, 6-10μ long, greenish-blue, pretty uniformly fine-granular.

Found on the bark of various trees and commonly on the white-washed plaster of the shady side of walls.

The specimen of Aegle Marmelos on the bark of which the alga was first discovered grows close to a building the roof of which had been recently reterraced; the calcium carbonate encrusting the algæ on other trees is possibly derived from the slaked line carted along the streets. The alga is never found without the coating of calcium carbonate.

Plate IV, fig. 15, a-f.

PORPHYROSIPHON NOTARISII (Meneghini) Kuetzing.

Stratum expansum, tomentosum, fusco-purpureum, set etiam, sucundum Kurz, initio plus minusve viride, aetate provecta colore cupreo; filamentis varie curvatis atque dense intricatis, cum vagina 10-20 μ crassis; vagina firma, lamellosa, apice in strata, plus minusve divergentia vel in fibrillas soluta, purpurea vel cuprea interdum aureofulva, apice saepe achroa, lamellis 2-6 vel pluribus trichomatibus aerugineis vel caeruleorviridibus, ad genicula vix vel manifeste constrictis, 4-6-8 (12-18) μ diametro; articulis 4-12 μ longis, diametro trichomatis subaequilongis vel eo usque ad triplum brevioribus; contentu granuloso; cellula apicali rotundata vel attenuata obtusaque.

Ad corticen arborum vel terram udam.

Stratum expanded, usually thin, densely and softly felty, purplish brown or copper-coloured when older; filaments closely interlaced, with the sheath 10-20 or more μ in diameter; sheath firm, lamellose, upwards resolved into loose layers or fibrils, purplish or copper-brown, sometimes golden-yellow, at the apex often colourless; trichomes verdigris-green or greenish-blue, not at all or distinctly constricted at the joints 4-12 (-18) μ in diameter; cellules 4-12 μ long, about equal in length to the thickness or only one-third as long; contents granular.

We have not found this species yet in Bengal, but it is not likely to be absent from this province, as it is practically cosmopolitan. We follow De Toni's treatise in including in this species the following: *Scytonema peguanum*, Martens, (Kurz' sheets 3139 3186), *Scytonema varium*, Martens (Kurz' sheet number 3241a), *Scytonema fulvum*, Zeller, (Kurz' sheet 314) and *Scytonema fuscum*, Zeller (Kurz' sheets 3153, 3187 and 3201), as after a careful examination of the Kurzian specimens preserved in the Sibpur herbarium we have not discovered any characters by which these forms can be specifically distinguished from *Porphyrosiphon Notarisii.* We also agree with Professor Achilles Forti in holding that it would be preferable to unite Porphyrosiphon with Lyngbya.

De Toni, Myxophyceae, p. 314. Gomont, Monographie des Oscillariacées, p 331, tab. 12.

Plate IV, fig. 16, a-c.

Scytonema ocellatum, Lyngbye.

Stratum e pulvillis 3-5 mm. diametro, circiter 1-2 mm crassis, plus minusve interruptis, siccitate nigris, humectatis badiis et subvelutinis compositum; filamentis 10-18 μ crassis, intricatis, pseudoramosis; pseudoramis brevibus, curvulis; vaginis firmis, haud lamellosis, fuscis, laevibus, 1.5—2 μ crassis; trichomatibus (6) 9-10 (—14) μ diametro, olivaceo-viridibus; articulis 4—8 μ longis, diametro saepissime brevioribus; contentu caeruleo—viridi, subgrosse granuloso; heterocystis transverse oblongis vel quadratis. 4-8 μ longis, 8-10 μ latis, homogeneis, flavidis.

Habitat ad corticem *Swieteniae Mahagoni*, *Mangiferae indicae*, *Tamarindi indicae*, *Polyalthiae longifoliae*, *Fici religiosae*, *Meliae Azidarachtae* in suburbibus Calcuttensibus et in dictione urbium Lucknow et Fyzabad ad corticem arborum variorum et in aliis locis.

Stratum consisting of small cushion-like aggregations 3-5 mm in diameter 1-2 mm. thick, more or less interrupted, usually quite black when dry, when fresh chocolate-brown and more or less velvety to the feel; filaments 10—18 μ in diameter, interwoven; false branches somewhat curved, short and rather scanty; sheaths firm, not lamellose, braun, smooth, 1·5—2 mm. thick; trichomata 6—12μ in diameter, olive-green; cellules 4—8 μ long, usually shorter than the diameter; contents somewhat coarsely granular; heterocysts square or short oblong, 8—10 μ wide, contents yellowish and homogoneous.

De Toni, Myxophyceae, p. 309.
Plate V, fig. 17, a-c.

Scytonema mirabile, (Dillwyn) Bornet.

Aut stratum pannosum, spongiosum, expansum, tomentosum, fusco-nigrum in terra uda tempore pluviarum, aut pulvillos 3—5 mm. diametro, plus minusve irregulares strato *Lyngbyae corticicolae* intersperso, nigros efficiens; filamentis tortuosis, intricatis, flexuosis, apicem versus paullulum attenuatis, apice obtusis, 15—24μ crassis; vagina lamellosa, 2—3μ crassa, fusca, lamellis 3—5, parallelis; trichomatibus 6—12, saepius circiter 10μ crassis, aerugineis, interdum flavido-aerugineis; articulis inferioribus diametro paullo brevioribus vel longioribus, 8—12μ longis, superioribus diametro distincte brevioribus,

3—8μ longis; contentu granuloso, granulis minutis atque grossis intermixtis, dense aggregatis, saepe dissepimenta obscurantibus.

In terra uda et ad corticem *Terminaliae Catappae* in horto botanico Sibpurensi. Lectum Julio 1922.

Forming either ragged extensive brownish or greenish black layers on wet ground or small cushions, 3—5 mm. in diameter on the bark of certain trees; filaments flexuous, interwoven, slightly narrowed down towards the apex, 15—24μ thick; sheath lamellose, 2—3μ thick, brown, lamellae 3—5, parallel; trichomes 6—12μ thick, verdigris-green, sometimes with a yellowish tint; lower cellules 8—12μ long, somewhat shorter or longer than the diameter, upper cellules 3—8μ long, distinctly shorter than the diameter; cell-contents granular, consisting mostly of fine granules intermixed with fewer very coarse granules, the finer granules often nearly hiding the dissapiments.

De Toni, Myxophyceae, p. 517. G. West. Algae, I, p. 44, fig. 33. Plate V, fig. 18, a-b.

SCYTONEMA TOLYPOTRICHOIDES, Kuetzing.

Thallus saepius natans, tum suborbicularis, 2—5 (—10) mm. diametro, sed etiam ad corticem *Terminaliae Catappae* maculas brunneas efficiens; filamentis a centro radiantibus, repetite pseudoramosis, 10—15μ crassis; pseudoramis strictis vel parelle sinuosis, filis primariis conformibus, raro singulis, saepissime geminatis, paribus pseudoramulorum approximatis; vagina 2—3μ crassa, fusca, lamellosa, apice fere achroa, lamellis oblique transversis, brevibus; trichomatibus 6—9μ crassis, caeruleo-viridibus; articulis 3—9μ longis, subquadratis vel oblongis; contentu dense granuloso; heterocystis 6—10μ longis, circiter 6μ latis, luteolis.

In horto botanico Sibpurensi. Lectum Augusto 1922.

The specimens were found near the base of a *Terminalia Catappa*, associated with *Lyngbya aerugineo-caerulea*, *Lyngbya corticicola* and *Lyngbya calcifera*, forming small circular brown patches about 2 mm. diameter; grown in water in a Petri dish most of them remained attached to the bottom of the dish, but a number of them floated on the surface and formed patches reaching a diameter of 5 mm.

The filaments radiate from the centre, and are 10—15μ thick; pseudo-branches straight or slighly curved, similar to the main branches, rarely single, nearly always in pairs, the pairs often rather

close together; vagina 2—3μ thick, consisting of short, more or less transversely placed lamellæ; trichomata 6—9μ thick, bluish green; cellules 3—9μ long, subquadrate or oblong; contents densely granular; heterocysts 6—10μ long, about 6μ wide, yellowish.

De Toni, Myxophyceæ, p. 916.
Kuetzing, Tab. Phyc. II, tab. 22.
Plate V. fig. 19, a-c.

SCYTONEMA ZELLERIANUM, Brühl et Biswas, sp. nova.

Among Kurz' collection of Indian algae preserved in the herbarium of the Royal Botanical Gardens, Sibpur, there are six sheets, namely numbers 1733, 1857, 2709, 3175, 3376, 3352, all of them marked *Scytonema aureum* Meneghini. A careful and detailed examination of all six specimens has convinced us that they belong to the same species. One of the specimens grew on a gneissic rock in the Choungmenah Hills, at an elevation of 2500 to 3000 feet; the second was found growing on rocks at Kayengmathay Choung in the Pegu Yomah; the third and fourth occurred on the bark of trees in Pegu (one at Elephant Point), the fifth (No. 1733) was collected in the Manbhum District and the sixth occurred on mud walls in the Sibpur Botanical Gardens. Three of the Burma specimens are referred to by G. Zeller in the journal of the Asiatic Seciety of Bengal, Vol. XLII, part II, p. 181 under the name of *Scytonema aureum*, Meneghini.

The following is a description based on the six Kurzian specimens.

Stratum pannosum, extensum, circiter 1 mm crassum, fuscum vel fusco-nigrescens, densissime tomentosum; filamentis elongatis, flexuosis, arcte intertextis, pseudoramosis; pseudoramis solitariis vel geminatis, elongatis, paullo tenuioribus, interdum recurvis; vagina firma, 3—5μ crassa, lamellosa, fusca, lamellis 2—5, parallelis; trichomatibus 12—20μ diametro, ad genicula haud constrictis, caeruleo-viridibus paullo flavidis; articulis 10—15μ longis, rarius subquadratis, saepius diametro brevibus; contentu uniformiter granuloso; heterocystis a pseudoramis remotis, quadratis vel oblongis, 15—20μ longis, 12—15μ diametro, colore luteolo.

Stratum extensive, about 1 mm thick, densely felty, from brown to nearly black; filaments long and flexible; false branches single or in pairs, slender, somewhat thinning out towards the apex, sometimes recurved; sheath firm, 3—5μ thick, consisting of 2—5 lamellae,

brown; trichomata 12—20μ thick, not constricted at the joints, bluish-green with a yellowish tint; cellules about as long as wide or usually shorter than long; contents uniformly granular; heterocysts distant from the base of the branches, 15—20μ long, square or oblong, yellowish.

Meneghini's *Scytonema aureum* was considered by Borzi to be a species of Tolypothrix, apparently after examination of more copious material; the description of *Tolypothrix aurea*, as given in De Toni's *Myxophyceæ* does not well agree with the Kurzian specimens; in any case the Kurzian and Zellerian species is certainly not a *Tolypothrix*, but is a *bona fide Scytonema*.

We have examined authenticated specimens of *Scytonemæ Myochorus*, Agardh, and are convinced that, notwithstanding some points of resemblance with that species, the present species is certainly different.

Plate V, fig. 20, a-d.

TOLYPOTHRIX BYSSOIDEA, (Hassall) Kirchner.

Stratum pulvinato-tomentosum, fuscum, nigrescens; filamentis plus minus erectis, cespitula efficientibus, irregulariter pseudoramosis, 10—15μ crassis; vagina tenui, 0·5—1·5μ crassa, fragili; trichomatibus torulosis, 8—9μ crassis; pseudoramis saepius brevibus, curvatis, interdum recurvis, patentibus vel erecto-patentibus, apicem sub-obtusum versus paullulo vel vix attenuatis, sub heterocystas vel ex heterocystis ipsis ortis; articulis 2—4μ crassis, transverse oblongis vel oblato-ellipsoideis, longitudine tertia pars latitudinis; contentu cellularum caeruleo-viridi, dense granuloso, granulis crassitudine varia; heterocystis sphaericis vel oblato-sphaeroideis, 10—12μ diametro.

Habitat ad corticem *Fici infectoriæ*, *Mangiferae indicae* et *Swieteniae Mahagoni* in suburbibus Calcuttensibus.

Lecta Augusto 1922.

De Toni, Myxophyceæ, p. 551.
Wolle, Freshw, Algae, U.S., pl. 181.
Cooke, Brit. Freshw. Algae, pl. 111.
Plate VI, fig. 21, a-c.

CHLOROPHYCEAE.

TRENTEPOHLIA TENUIS (Zeller sub nomine *Chroolepus*] *tenue*, in Journal of the Asiatic Society, Vol. XLII, Part II, p. 191. Sheet number 3268 of Kurz' Collection, preserved in the Herbarium of the Royal Botanic Gardens, Sibpur.)

Stratum e cespitulis exiguis arcte aggregatis compositum, aurantiacum (e notula Kurzii), in siccitate aurantiaco-cinereum; filamentis primariis subrectis vel flexuosis; filamentis ordinum superiorum varie divaricatis vel recurvis; cellulis filamentorum primariorum plerumque ellipsoideis vel ellipsoideo-oblongis, rarius subglobosis, 6—8μ diametro, saepius (8—) 10—14μ longis; cellulis ramorum 4—6μ diametro, haud raro prope basin subisodiametricis, plurimis ellipsoideis; cellulis omnibus ad genicula manifeste constrictis, filamenta torulosa efficientibus, saepe plus minusve asymmetricis; membrana cellulari hyalina, 1—7·5μ crassa, superficie externa asperrima, quasi granulata; contentu granuloso; gonidiis globosis, terminalibus et lateralibus.

Habitat ad corticem *Sonneratiae apetalae* in ditione Pegu.

Stratum consisting of minute closely packed tufts, orange-coloured when fresh, ashy-grey with a dull-orange tint when dry. Filaments nearly straight or flexuous, those of higher orders divaricate in various degrees. Cells always distinctly constricted at the dissepiments, elongate-ellipsoidal or short-subcylindrical, often asymmetric, those of primary and secondary filaments (8—) 10—14μ long, 6—8μ at the widest, rarely as long as wide, usually 1½—twice as long; those of the thinner filaments 4—6μ at the widest. Cell-wall 1—1·5μ thick, very rough on the outer surface. Contents granular. Gonidia spherical, terminal or lateral, not clustered.

Plate VI, fig. 22, a-d.

TRENTEPOHLIA AUREA (Lin.) Martens, *var.* TENUIOR, Brühl et Biswas.

Stratum tenue, substrato arcte appositum, extensum, colore aureo vel aurantiaco, in siccitate plus minusve cinerascente, densissime tomentosum aut e pulvillis parvis compositum; filamentis prostratis atque ascendentibus, saepe valde elongatis, plus minusve ramosis; cellulis vegetativis cylindricis, medio saepe leviter irregulariterque intumescentibus, ad dissepimenta vix vel haud constrictis, 14—24μ

longis, 4—8μ crassis (longitudo : crassitudo : : 12 : 4, 14 : 6, 15 : 6, 16 : 6, 18 : 6, 21 : 8, 24 : 8μ), terminalibus apicem obtusum versus saepissime attenuatis; contentu e guttulis intense coloratis et creberrimis atque granulis chlorophyllaceis paucis protoplasmateque composito; membrana cellularum vulgo e lamellis tribus composita, cellulae terminalis apice crassiori; zoogonidangiis lateralibus vel terminalibus, sphaericis, circiter 16μ diametro aut obovoideis, 12—20μ longis et 9—16μ crassis; zoosporis biciliatis.

Stratum thin, extensive, when fresh golden-yellow or orange-coloured, when dry more ash-coloured, densely felty or consisting of minute closely packed cushions. Filaments at first prostrate, further on more or less ascendant, often elongate, more or less branched; vegetative cells cylindrical, in the middle often slightly and irregularly swollen, at the joints slightly or not at all constricted, 4—8μ thick, 14—24μ long, the terminal ones narrowed towards the apex; contents consisting of protoplasm, nucleus, scanty chlorophyll granules and crowded droplets coloured by carotinoids. Cell-membrane consisting commonly of three lamellae; zoogonidangia lateral and terminal, either spherical and about 16μ in diameter or obovoid, 12-20μ long, 9-16μ wide. Zoospores biciliate.

According to De Wildeman, Heering and Fischer the vegetative cells of the typical variety are 9—24 (—30)μ wide and one and a half to three times as long, and the zoogonidangia (9—) 18—38μ in diameter.

Found on the bark of *Terminalia Arjuna* in the Royal Botanical Gardens, Sibpur.

See Annales du Jardin Botanique de Buitenzorg, 1897.
I. Supplement. Also vol. IX, p. 131.
Oesterrichische Botanische Zeitung, 1922, p. 19.
De Toni, Syll. Alg. I, p. 236.
Wolle, Freshw. Alg. U.S., pl. 115.
Plate VI, fig. 23, a-e.

TRENTEPOHLIA GRACILIS, P. Iyengar, *sp. nova.*

Stratum tenue, continuum, plus minusve viride vel luteo-viride, interdum fulvum; filamentis elongatis, parte basali prostrata, ramis vix crebris, modice flexuosis, cum parte primaria angulum acutum vel fere rectum efficientibus; cellulis partis prostratae saepe subglobosis, sed etiam ellipsoideis, ad dissepimenta manifeste, rarius leviter con-

strictis; cellulis ramorum ellipsoideis, etiam pyriformibus, saepe plus minusve asymmetricis; cellulis terminalibus subglobosis vel ellipsoideo-oblongis vel sublageniformibus; crassitudine cellularum 3—10μ, longitudine crassitudini fere aequali vel duplo crassitudinis; membrana achroa, tenui, laevi, circiter 0.5μ crassa; contentu granuloso, cum stillis sphaericis achrois vel pallide luteis singulis vel paucis, in cellulis basalibus manifestioribus; nucleo sphaerico, subcentrali.

Layer thin, continuous, of greenish, yellowish-green, more rarely brown tints. Filaments elongate, prostrate below, branches somewhat remote from each other, moderately flexuous, forming an acute or nearly right angle with the main filament. Cellules of the prostrate part often subglobose, but also ellipsoidal, at the joints conspicuously, more rarely slightly constricted; cellules of the branches ellipsoidal, sometimes even pear-shaped, often more or less asymmetric; terminal cells subglobose, ellipsoidal-oblong or somewhat flask-shaped; diameter of cells 3—10μ, about as long or twice as long as thick; cell-membrane colourless, thin, smooth; cell-contents granular and with single or several globular, colourless or pale-yellow droplets, which are more prominent in the basal cells; nucleus spherical, subcentrally placed.

Trentepohlia gracilis forms a thin coating on the bark of climbing plants (Aristolochia and others), as well as on iron posts, railings, tee irons and twisted galvanic iron wires in the Orchid House of the Botanical Garden in Sibpore. The coating is, as a rule, continuous except where it sits on flakes of iron rust. Near the ground the colour of the coating is nearly a pure green, higher up it merges into different shades of yellowish-green, whilst near the top it assumes more decidedly orange tints. When moistened after having dried up it absorbs water greedily and assumes a greenish black colour. The droplets mentioned above are oil droplets coloured by carotin, which gives the more orange-coloured cells their tint; minute chlorophyll granules usually predominate. We have not observed any zoospores, and therefore we are not sure about the larger spherical cells being zoosporangia.

The present species evidently belongs to the moniliform group of Trentepohlias. *Trentepohlia moniliformis* of Karsten (the *T. Monile* of Wildeman and the *T. monilia* of De Toni) agrees as to the shape of the cells, the thinness and smoothness of the cell-wall and the relative dimensions of width to length; but the length of the cells of *T. moniliformis*, according to Wildeman, is 14—23μ, according to Karsten 30μ,

the greatest width being 20μ, whilst in *T. gracilis* the largest dimension is 10μ.

Of the more or less moniliform species of *Trentepohlia* the width of which lies between 5 and 10μ, *T. abietina* and *T. procumbens* are distinguished by the common occurrence of clusters of sporangia. *Chroolepus tenue* of Zeller, described in the present paper under the name of *Trentepohlia tenuis*, is easily distinguishable by its rough cell-walls and other characters.

Plate VII, fig. 24, a-f.

TRENTEPOHLIA TORULOSA, Wildeman.

Stratum tenue, corticem arborum vestiens, extensum, subuniforme vel saepius quasi granulosum, e pulvillis vel caespitulis minimis arcte confertis compositum, in siccitate luteo-virescens vel fulvum vel ochraceum; filamentis torulosis, primariis saepe elongatis, ramosis, ramis ramulisque patentibus, pauci—aut pluricellularibus; cellulis vegetativis ellipsoideis, raro subsphaericis, ad dissepimenta manifeste constrictis, interdum symmetricis, sed saepissime plus minusve conspicue asymmetricis, latere altero modice convexo, altero ventricoso, dimensionibus (longitudo × latitudo): 36 × 20, 32 × 18, 30 × 12—16, 28 × 18, 26 × 16, 22 × 16, 20 × 16, 20 × 10, 18 × 18, 16 × 14μ, ad dissepimenta 4—10μ latis; membrana cellularum saepissime tenui, 1—$1{\cdot}5\mu$ crassa, vulgo laevi, interdum minutissime quasi muricatula; contentu granuloso composito e protoplasmate atque guttulis carotinoideis coloratis, guttulis saepe prope partem ventricosam aggregatis, granulis chlorophyllaceis viridibus, interdum in discos raro in taeniam aggregatis saepissime singulis intermixtis; gonidangiis sphaericis, raro intercalaribus, saepissime lateralibus vel terminalibus, 14—24μ diametro, aut sessilibus aut terminalibus pedicellatis, cellula pedicellari inferne ovoideo, hic circiter 20μ lata, superne uncinatim curvata et circiter 10μ diametro ("Hakensporangia").

Habitat ad corticem *Swieteniae Mahagoni*, *Mimusopis Elengi* et *Poincianae regiae* in horto botanico Sibpurensi et in suburbibus Calcuttensibus.

The Bengal form agrees with those reported from Sumatra in possessing hook-supported as well as sessile globose sporangia, but differs in the cell-walls always being thin, not thick ("membrane épaisse" of Wildeman). *Trentepohlia moniliforme* of Karsten, which according to this author has cells about 30μ long and 20μ at

their widest, possesses a *thin* cell-membrane. Unfortunately De Wildeman did not give the thickness of the cell-walls of the two species in microns and his figures do not show any difference in this respect. *T. moniliformis* is said to be common in Java. As other species of *Trentepohlia* with habitually thick cell-walls occur with thin walls either in the younger stages of their development or in damp localities or in other conditions, it is probable that the relative thickness of the cell-walls cannot be employed to distinguish *T. torulosa* and *T. moniliformis*, Karsten, from each other. As regards nomenclature, Kuetzing pictures under the name of *Chroolepus moniliforme* what undoubtedly is *Trentepohlia umbrina* surrounded by fungal hyphae and the name has therefore to be dropped as having been erroneously applied to a composite form, but that seems to be not a sufficient reason why Karsten's name should not stand. De Wildeman changed Karsten's name into *Trentepohlia Monile*, where *Monile* is evidently meant to be a noun, signifying "necklace"; for "monile" cannot be an adjective associated with *Trentepohlia*, the latter being feminine, whilst *monile* would be neutre. De Toni changed *monile* into *monilia*, a form ultimately adopted by Wildeman. Unfortunately *monilius* is rather impossible as a Latin adjective derived from *monile*. Scarcely any confusion can arise from adopting Karsten's name *Trentepohlia moniliformis* for the species referred to. As we cannot find, in the literature at our disposal, any statement about the sporangia of *T. moniliformis*, Karsten, and as the sporangia of the Bengal species agree with those described as occurring in *Trentepholia torulosa*, Wildeman, we have adopted the latter name for the Indian bark alga described above in detail.

See Wildeman, Les Trentepohlia des Indes Néerlandaises in Ann Jard. Bot. de Buitenzorg, vol. IX, p. 139.

Plate VII, fig. 25, a-f.

BOTANICAL LABORATORY,

UNIVERSITY COLLEGE OF SCIENCE,

BALIGANJ,

The 1st April, 1923.

EXPLANATION OF FIGURES.

Plate I.

Fig. 1. *Chroococcus turgidus*, × 950.

Fig. 2. *Chroococcus minutus*, × 600.

Fig. 3. *Aphanocapsa brunnea*, × 500.

Fig. 4. *Microcystic marginata*, × 650.

Fig. 5. *Oscillatoria Kuetzingiana*:

(*a*) bundle of filaments; × 1000;

(*b*) single filament, × 1500.

Fig. 6. *Oscillatoria Acula*:

(*a*) bundle of filaments, × 550;

(*b*) single filament, × 700.

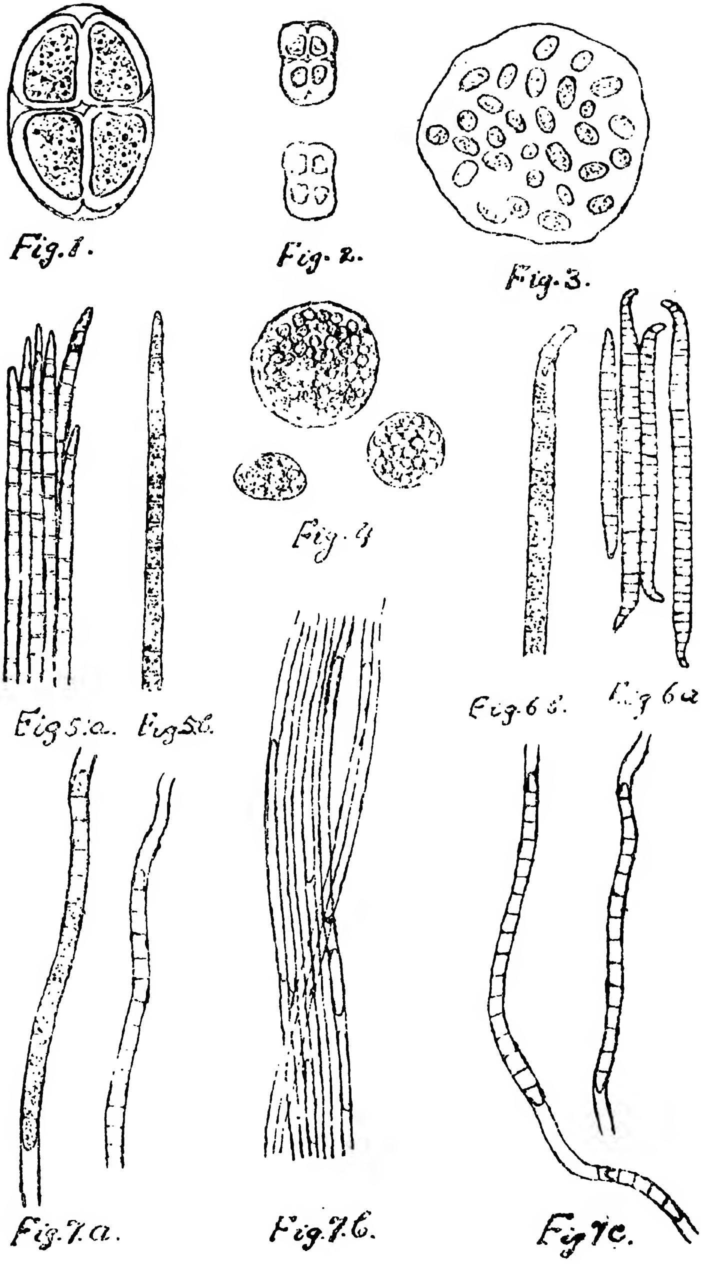
Fig. 7. *Lyngbya subtilis*:

(*a*) *var. granulosa*, single filament, × 1000.

(*b*) the same, bundle of filaments, × 500;

(*c*) *var. typica*, × 1000.

PLATE I



Drawn by K. P. Biswas.

EXPLANATION OF FIGURES.

Plate II.

Fig. 8. *Lyngbya connectens*:

(*a*) stratum, × 50;

(*b*) bundle of filaments with hormogones, × 350;

(*c*) portion of filament, × 600;

(*d*) part of filament with escaping hormogone, × 600;

(*e*) escaped trichome, showing apical cell, × 600.

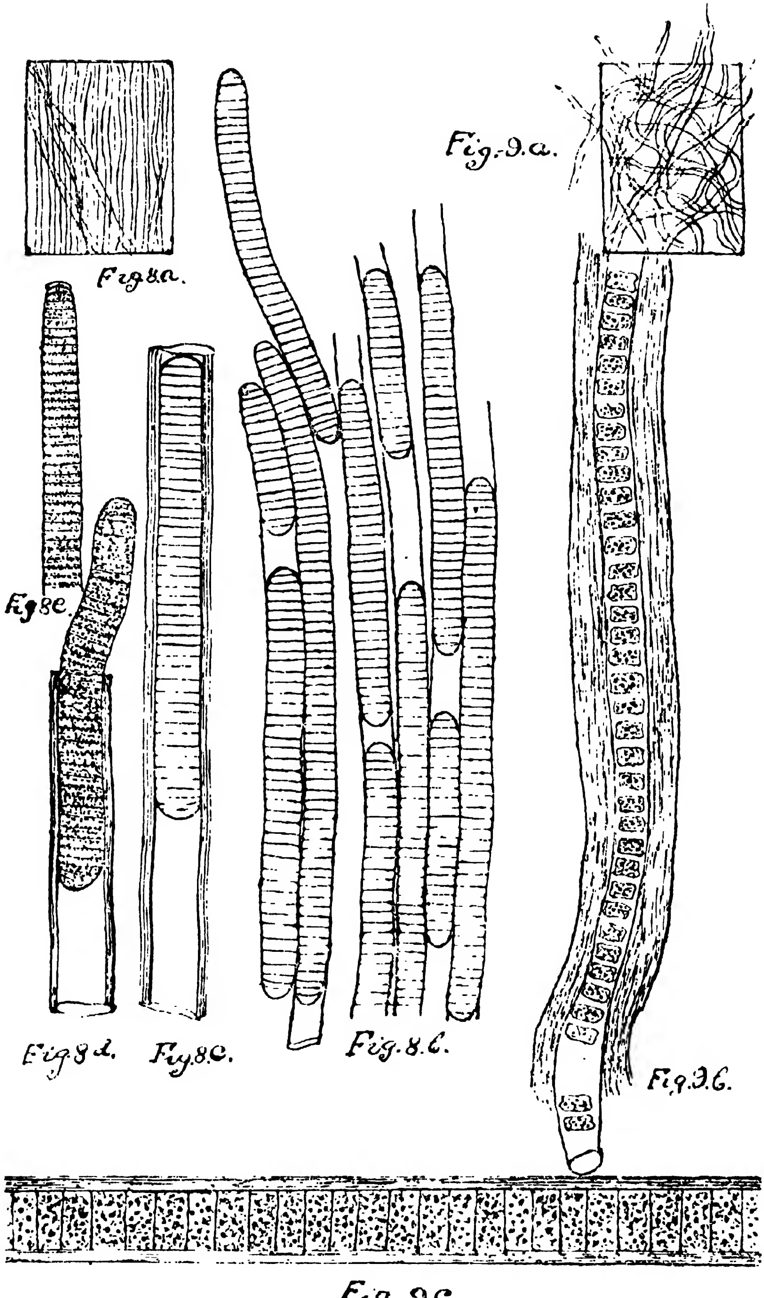
Fig. 9. *Lyngbya aestuarii, var. arbustiva*:

(*a*) stratum, × 50;

(*b*) part of filament, glycerine preparation, × 600;

(*c*) part of filament, living, mounted in water, × 500.

PLATE II



Drawn by K. P. Biswas.

EXPLANATION OF FIGURES.

PLATE III.

Fig. 10. *Lyngbya arboricola* :

(*a*) stratum, × 50 ;

(*b*) part of old filament, × 650 ;

(*c*) part of younger filament, × 650 ;

(*d*) empty sheath, × 650 ;

(*e*) hormogones.

Fig. 11. *Lyngbya dendrobia* :

(*a*) stratum, × 50 ;

(*b*) three filaments, × 350 ;

(*c*) younger filament, × 700 ;

(*d*) older filament, × 700 ;

(*e*) apex of younger filament, × 800 ;

(*f*) hormogones.

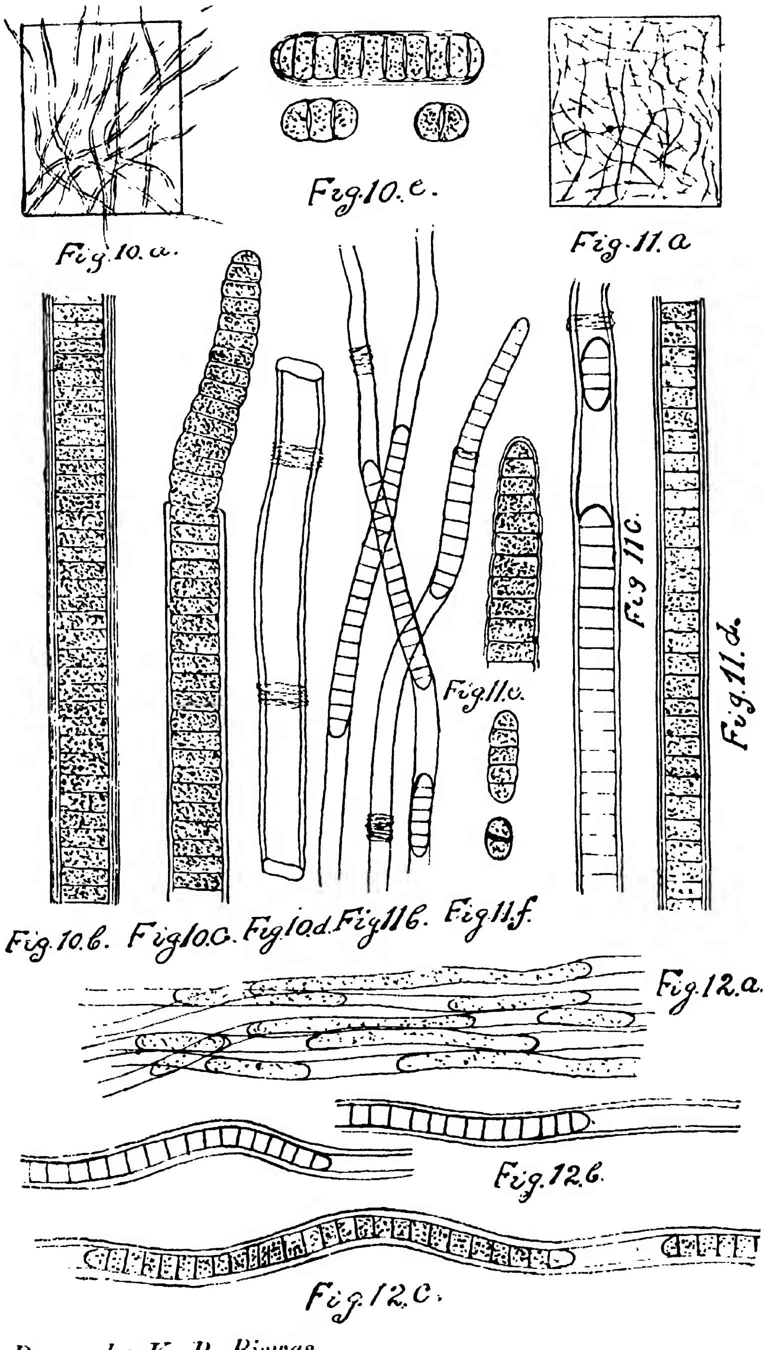
Fig. 12. *Lyngbya aerugineo-coerulea* :

(*a*) bundle of filaments, × 500 ;

(*b*) parts of two filaments, × 1000 ;

(*c*) single filament, × 1000.

PLATE III



Drawn by K. P. Biswas.

EXPLANATION OF FIGURES

PLATE IV.

Fig. 13. *Lyngbya corticicola* :

(*a*) stratum, × 50 ;

(*b*) upper part of filament, × 700 ;

(*c*) a whole filament, × 500 ; dissepiments obscure ;

(*d*) hormogones, about 500.

Fig. 14. *Lyngbya palmarum* :

(*a*) stratum ;

(*b*) group of filaments, × 350 ;

(*c*) part of filament, × 1000.

Fig. 15. *Lyngbya calcifera* :

(*a*) stratum ;

(*b*) stratum, more highly magnified ;

(*c*) part of filament, after treatment with the dilute hydrochloric acid, × 800 ;

(*d*) escaped trichome, × 400 ;

(*e*) part of filament with calcareous deposit, × 800 ;

(*f*) part of filament with hormogones, × 800.

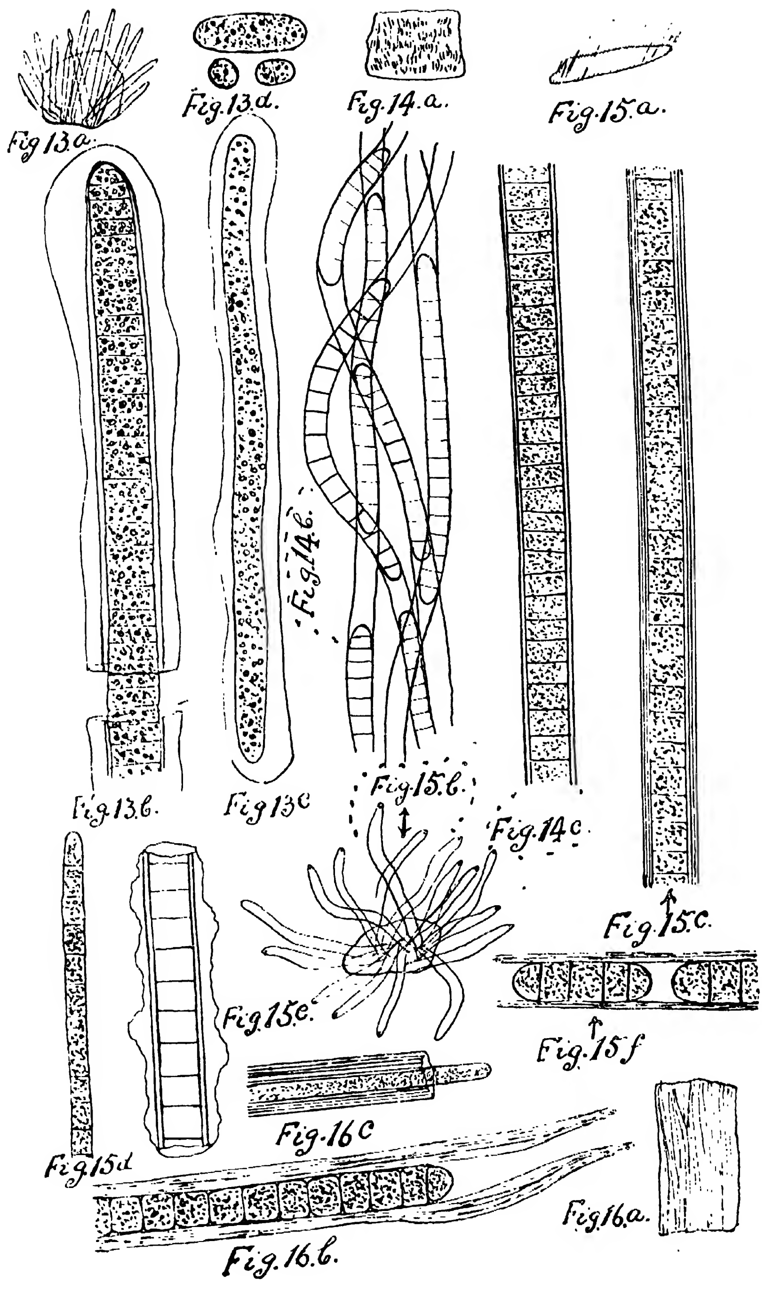
Fig. 16. *Porphyrosiphon Notarisii* :

(*a*) stratum ;

(*b*) upper part or filament, × 1000 ;

(*c*) part of sheath with escaping trichome, × 400.

PLATE IV



Drawn by K. P. Biswas.

EXPLANATION OF FIGURES.

PLATE V.

Fig. 17. *Scytonema ocellatum*;

(*a*) stratum;

(*b*) part of filament with heterocysts and young pseudobranch, contents omitted, × 500;

(*c*) part of branched filament, contents fully depicted, × 500.

Fig. 18. *Scytonema mirabile*;

(*a*) stratum;

(*b*) part of filament with pseudobranches, × 350.

Fig. 19. *Scytonema tolypotrichoides*:

(*a*) stratum;

(*b*) filament with pseudobranches, contents omitted, × 150;

(*c*) upper part of filament, showing contents and obliquely stratified sheath, × 750.

Fig. 20. *Scytonema Zellerianum*:

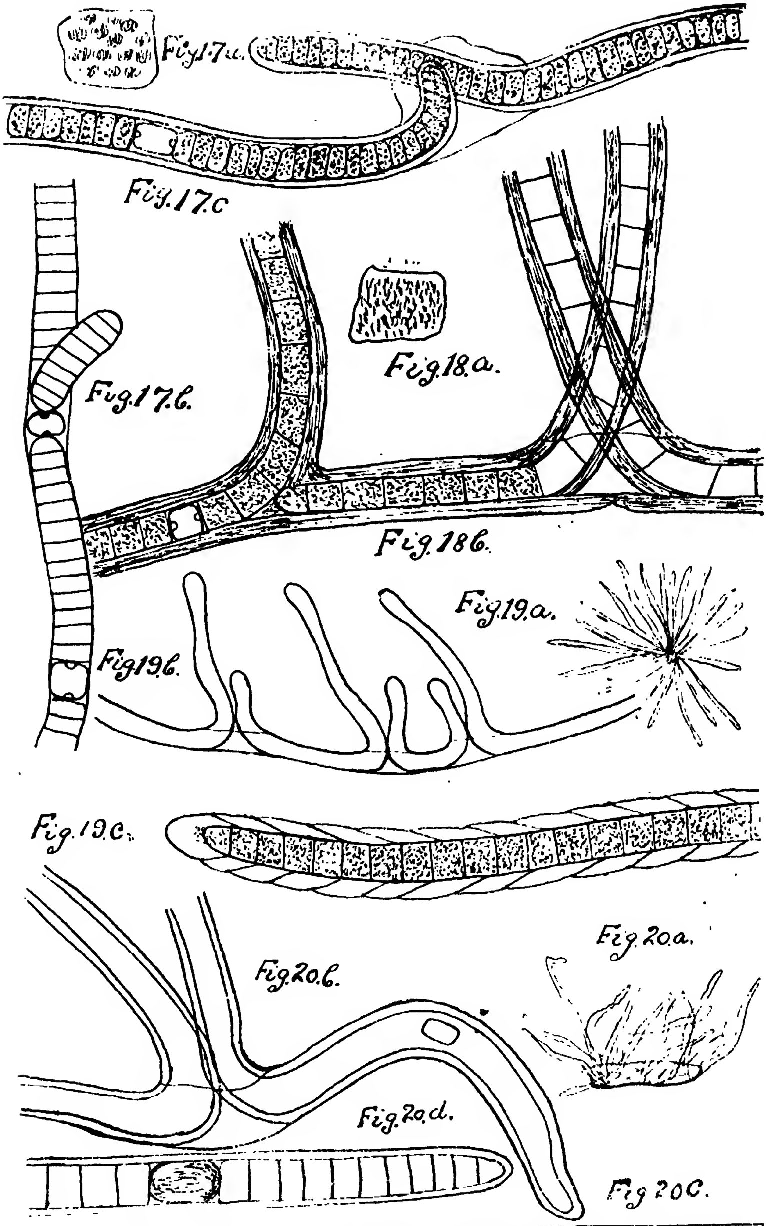
(*a*) stratum;

(*b*) part of filament with two pseudobranches, contents not shown, × 350;

(*c*) part of filament with heterocyst, contents and lamellar sheath shown, × 350:

(*d*) apex of filament, × 350.

PLATE V



Drawn by K. P. Biswas.

EXPLANATION OF FIGURES.

Plate VI.

Fig. 21. *Tolypothrix byssoidea*:

(*a*) stratum;

(*b*) branched filament with heterocysts shown, × 250;

(*c*) part of filament with pseudobranches and heterocysts, contents shown, × 500.

Fig. 22. *Trentepohlia tenuis*:

(*a*) part of filament, contents omitted, × 1000;

(*b*) similar, contents shown;

(*c*) filament with lateral sporangium, × 650;

(*d*) basal cells, × 1000.

Fig. 23. *Trentepohlia aurea, var. tenuior*:

(*a*) branched filament, × 400;

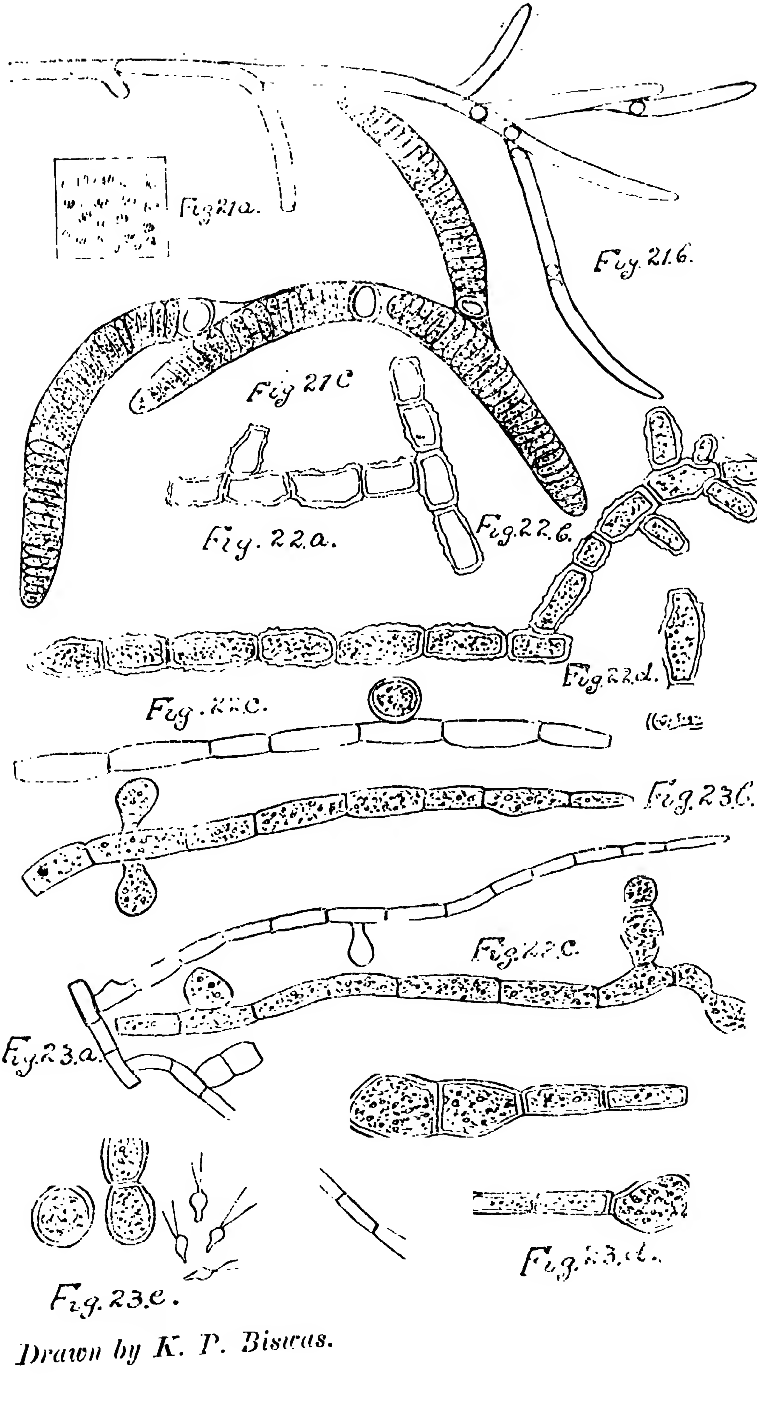
(*b*) part of filament with two lateral sporangia, × 800;

(*c*) part of filament with lateral and terminal sporangia;

(*d*) apex of filaments with terminal sporangia;

(*e*) sporangia and zoospores.

PLATE VI



Drawn by K. P. Biswas.

EXPLANATION OF FIGURES.

Plate VII.

Fig. 24. *Trentepohlia gracilis*:

(*a*) stratum, × 200;

(*b*) branched filament, × 500;

(*c*) branched filament, cell contents shown, × 500;

(*d*) basal part, with three branches, × 500;

(*e*) parts of filament, showing apical cell and sporangia, × 500;

(*f*) basal cell with young branch, × 500.

Fig. 25. *Trentepohlia torulosa*:

(*a*) much-branched filament, × 500;

(*b*) branched filament with supporting cells of zoogonidangia, × 500;

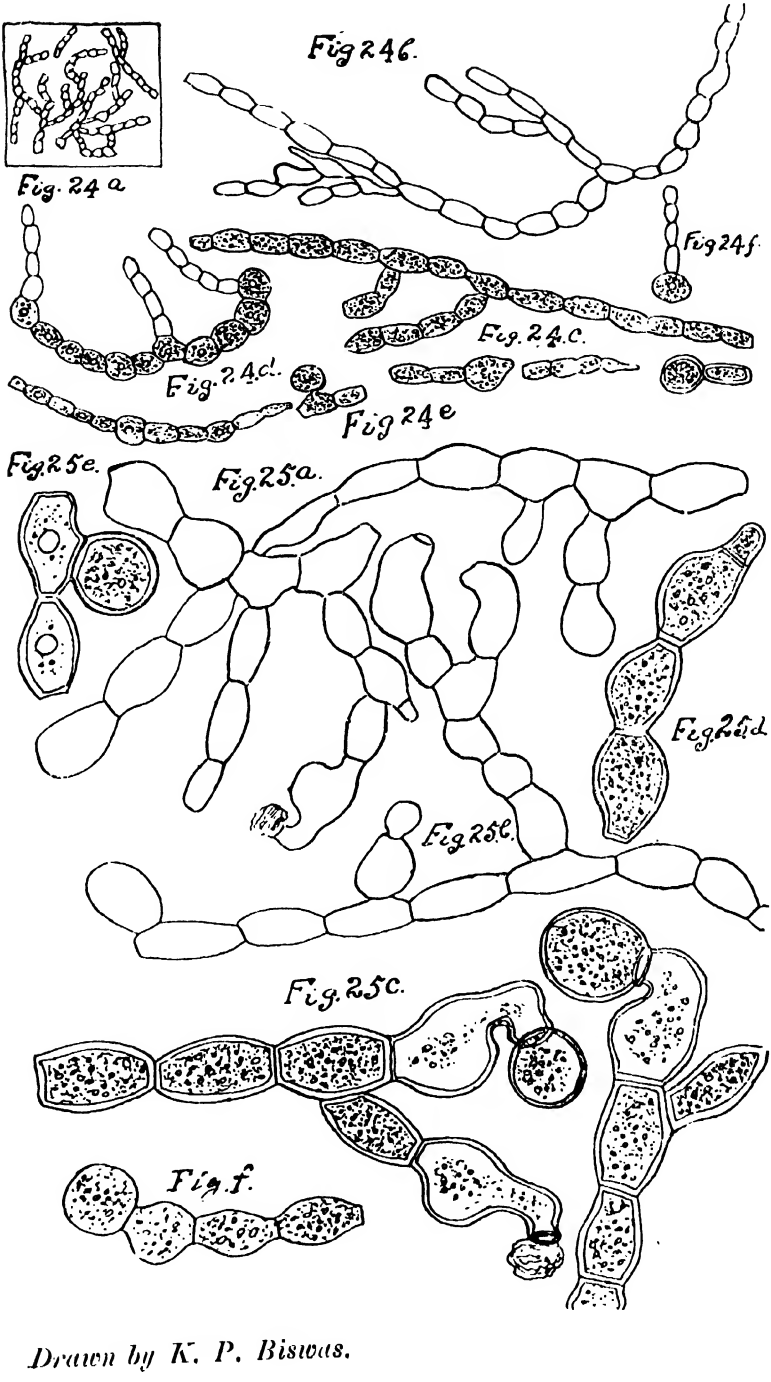
(*c*) parts of filaments with sporangia and supporting cells, × 600;

(*d*) upper part of filament with apical cell, × 600;

(*e*) part of filament with lateral zoosporangium, treated with chlorozinc iodide, × 600;

(*f*) apical part of filament with young zoosporangium and supporting cell, × 500.

PLATE VII



Drawn by K. P. Biswas.

COMMENTATIONES ALGOLOGICAE

III

ON A SPECIES OF COMPSOPOGON GROWING IN BENGAL

BY

P. BRÜHL, D.SC., AND KALIPADA BISWAS, M.A.

In December, 1920, we observed a much-branched alga growing on one of the brick steps leading down to the pond in the grounds of the Biological Laboratories, University College of Science, Baliganj, Calcutta. A specimen of the alga was kept alive in Knop's culture solution for several months in the laboratory, but finally died. On examination it proved to be an undoubted species of *Compsopogon*, showing all the essential features of the *Compsopogon* depicted on pag 319 of Part I, section 2 of Engler and Prantl's Pflanzenfamilien. It was only later on that our attention was drawn by N. Svedelius' note on the Compsopogonaceae, in Engler and Prantl's Nachtrag zum I. Teil, 2. Abteilung, to the highly interesting paper by R. THAXTER entitled "Note on the structure and reproduction of Compsopogon" (Botanical Gazette, Vol. XXIX, 1900).

In December, 1921, we observed a number of young filaments of the same species of *Compsopogon* growing on stems and roots of certain grasses and on the stems and leaves of *Hydrilla verticillata* in a pond within the grounds of the nursery belonging to Mr. A. Bose in Baliganj. In January following the alga was found growing in bunches all along the edge of the nursery pond, occurring not only attached to the submerged parts of aquatic plants, but also on the mud below the level of the water. Whilst the Compsopogon grew in profusion in the pond of nursery referred to, we were unable to discover the slightest trace of it in the College pond. In January of the present year, however, we observed the species again both in the pond of the nursery as also that in the College grounds. It is noticeable that the

plants begin to disappear as soon as the hot season sets in, so that to find specimens of them one has to look for them in the months of December, January and February. The disappearance does not depend entirely on the ponds being cleared of weeds under orders of the Calcutta Municipality, although its date is influenced by that operation. Quite lately we discovered very healthy specimens in a small pond beyond the Baliganj-Sealdah railway line; further investigation will probably show the species to be pretty widely distributed in Lower Bengal. Sir David Prain in Volume III, No. 2, of the Records of the Botanical Survey of India, in which he deals with the vegetation of the Districts of Hughli-Howrah and the 24-Pargannas, states that *Compsopogon Hookeri*, Montagne, occurs at the canal banks near the Calcutta Salt Lakes and in the marshes near Matla. This form may be identical with the species observed by us in and near Baliganj, but we have not yet been able to spot it at the localities mentioned by Sir David Prain.

The following is a description based on the specimens observed by us.

Young specimens are always found attached to stems, branches, or leaves of aquatic phanerogams as well as to small submerged sticks, decaying leaves, or bricks. They form comparatively short filaments attenuated at both ends, attached by a slightly broader base and consisting of a single row of cells (*see* fig. 5, plate IX). Very soon much smaller cells are cut off from the main row of cells by periclinal cell-walls so as to form a continuous rind (*see* fig. 6, Plate IX).

After the simple undivided young individuals have attained a certain length, they begin to give rise to lateral branches, which form angles with the parent stem varying commonly between about 30 and 60 degrees, the angle being usually not very different from 45 degrees. Branches of the first order branch again similarly, and so on to branches of higher orders. As far as our own observations go, the branches always arise from cells of the axial row before the formation of rind-cells; we have never observed branchlet arising from undoubted rind-cells.

The primary filaments as well as the branches and branchlets, as long as they consist of a single row of cells, are distinctly constricted at the joints; their cells are discoid, always wider than long, with rounded side-walls, the length of cells which have not recently divided by the formation of their transverse walls being pretty uniform, whilst

the width depends on the position of the cell in the branchlet; it will be noticed that growth in length takes place probably always by intercalary transverse division. (*See*, for instance, fig. 21 on plate X.) The cell-walls of fully formed cells are colourless, firm and about 3μ thick. The cells contain numerous chromatophores, which are always spheroidal or ellipsoidal, never anvil-shaped, nor sigmoid, nor resembling the "ossula phalangarum supremarum;" they are more densely aggregated close to the cell-wall, sometimes arranged in rows, but always discrete, never "pectinatim compacta."

Sooner or later, in the case of the primary filament and the lower primary branches at an early stage (*see* for instance figure 6 on plate IX), by the formation of cell-walls placed more or less obliquely to the axis of the filament, a ring of cells is cut off from each of the original cells; as the latter grow in length and breadth, the peripheral cells divide by the formation of cell-walls orientated in different directions, most of them obliquely to the axis, the result being a cortical layer of very small cells, irregularly polygonal in face-view; the greatest and least dimensions of these rind-cells were found to be in surface view 12-9, 12-12, 9-6, 6-6, 6-4, 6-3μ. The rind-cells may form only a single layer, but by further tangential divisions, which do not seem to follow any definite rule, the rind may consist locally of as much as five cells arranged radially. This is especially well seen in figure 25 of plate XI, which represents a cross-section through the basal part of an older plant of very irregular outline.

The axial column, formed by whatever is left after the cutting-off of the cortex, consists of a tissue of thin-walled cells, most of them of considerably larger size than the rind-cells and divided into sets by the much expanded original transverse cell-walls, which retain their greater thickness and firmness and which can usually be more or less distinctly seen in transmitted light, except where they are obscured by thicker portions of the cortex. This is brought out by various figures on plates IX and X. The walls of the cells forming the central column are exceedingly delicate and probably often tear during the process of growth. If, however, microtome sections are prepared with the utmost care, these delicate cells can be seen even in sections through the oldest parts of the main filament (*see* the figures on plate XI, and also figures 22 and 23 on plate X).

Although we have examined numerous, especially younger, specimens attached to sticks or leaf-blades, we have not discovered

any evidence of the formation of "rhizines." The young plant is attached by a cell which is wider than those immediately succeeding it in an upward direction and which, like the latter ones, surrounds itself later on with a ring of rind-cells; this ring, together with the axial cells, forms a broader base, which gives the growing plant a firm hold; we may call it the foot-plate (*see* figures 5 and 6 on plate IX).

The axial column of cells is of so delicate a nature that it can on no account be considered to form a "skeleton;" the mechanical strength of the thicker parts of the plant is entirely due to the rind-cells which are stayed by the firmer cross-walls of the original cells. The short branchlets of higher orders owe their sufficient mechanical strength to the comparatively firm walls of their constituent cells. If the axial column at all adds to the mechanical strength of the filaments, that can only be due to the cells being tightly filled with what is probably nearly pure water.

The colour of the plants varies considerably. A dense tuft growing near the edge of the nursery pond was dark verdigris-green in the middle portion, nearly black lower down and deep buff coloured where more freely exposed to the sun's rays. When cultivated in glass jars, either in ordinary pond water or in Knop's solution, the plant soon loses its original colour and assumes a pale buff tint. In some localities the plant, and more especially the younger branches and branchlets are decidedly purple or rose-coloured; but the normal tint may be taken as verdigris-green or greenish-blue.

REPRODUCTION.—The plant produced both micro-aplanospores and macro-aplanospores, exactly as described by Prof. Ronald Thaxter in his "Note on the Structure and Reproduction of Compsopogon" (Botanical Gazette, Vol. XXIX, 1900). The microsporangia occur in sori formed from single cells or small groups of cells forming constituent parts of the cortex (*see* figs. 10, 11, 12, 15 and 16 on plate X). The sori are more or less hemispherical or oblong in shape and consist of 15 to 25 or even more microsporangia; the latter are normally spherical, but very commonly polyhedral, due to mutual pressure. They may also arise from any cells of the filaments before the formation of the rind. In any case, the sori may be formed anywhere on the branches, so that ultimately they appear as scattered minute heaps of cells. The diameters of the hemispherical sori, as measured, were 25, 40, 50, 65, 75μ; the oblong sori were 100×50, $90 \times 60\mu$. The dimensions of the microsporangia, in surface view, were found to be 12×9,

12×12, 9×6, 6×4, $6 \times 3\mu$; the spherical ones were 6 to 12μ in diameter. In the formation of the micro-aplanospores the contents of the microsporangia contract slightly and become spherical. The wall of a microsporangium opens by a slit and the microspore is discharged with a jerk, the impetus given to it being evidently due to the turgescence of the surrounding, yet intact, microsporangia. The microspore is thus projected to some distance from the parent sporangium and finally floats away out of the field of view. It is $3 \cdot 4\mu$ in diameter, contains a central nucleus and a somewhat denser peripheral shell of chromatophores, the latter being less densely crowded nearer the centre; the colour of the microspores is pale bluish-grey.

The formation of the macrospores has been described by Professor Thaxter in great detail; the process can be easily inferred from figures 18*a* and 18*b* on plate X. The following is an account of observations made in the botanical laboratory of the Science College, Baliganj.[1] A bunch of the plant was kept hanging in a cylinder filled with water and a branch was observed under the microscope. A considerable number of macrosporangia had formed during the preceding day. They contained ellipsoidal chromatophores densely crowded not only close to the cell-wall, but also in the central part round the conspicuous nucleus. At about midnight the macrospores commenced to escape. After the bursting of the cell-wall the contents of the macrosporangium escape, but not with a jerking, but a gliding motion. The macroaplanospore thus liberated is seen to be enclosed in a slimy envelope, which, however, dissolves within thirty to sixty seconds, after which it floats out of the field of view, but very much less rapidly than the microspores. The macrospore is spherical in shape, $6\text{-}10\mu$ in diameter, has a centrally placed nucleus and closely packed ellipsoidal bluish-grey chromatophores.

It may be added that the macrosporangia are found both on the primary row of cells constituting younger branches as also on the cortex of older branches. They appear to be formed in large profusion during night-time, but can also be observed, though in much smaller numbers, during the day, and the escape of macropores is not entirely confined to the hours of the night.

Professor J. Bapt. De-Toni in Vol. IV of his invaluable "Sylloge Algarum," Florideae, Sectio I, pp. 28-30, describes seven species of

[1] The observations described in the following lines were made by my collaborator, Mr. Kalipada Biswas, between 11 o'clock and 5 o'clock at night on the 13th to the 14th February, the 13th being the fasting day of the Shivaratra festival. P.B.

Compsopogon, of which five are depicted in Kuetzing's Tabulae Phycologicæ, Volume VII, plates 88 and 89. Of the species described by De-Toni *Compsopogon fuscatus*, Zanardelli, a small species not exceeding 25 mm. in height, covering rocks on the island of Borneo with a yellowish-brown felt may be at once excluded ; it may not even be a species of Compsopogon. *C. Corinaldii*, Kuetzing, appears to be hardly known ; Kuetzing's illustration of it does not suggest specific identity with our plant. *C. leptocladus*, Montagne, of which an excellent illustration is found in Part I, Section 2, p. 319 of Engler and Prantl's Pflanzenfamilien, has a much thinner main stem and branches and is said to possess anvil-shaped or dumb-bell-shaped chromatophores (endochromata). *C. chalybeus*, Kuetzing, is stated to have erect and strict branches and, like *C. aeruginosus*, has very much smaller rind-cells than our species. *C. lividus* (Hooker), which has been reported from Madras, is stated to have complanate fronds and "patentissima" branches; this does not at all agree with our species. There remains *C. coeruleus*, Montagne. Specimens of the alga collected by Prof. Thaxter in Florida were compared by Bornet with specimens of *C. coeruleus* from Porto Rico in Montagne's herbarium and are declared to be "beyond question identical with Montagne's type specimens. In De Toni's Sylloge the chromatophores are stated to be "pectinatim compacta transversim oblonga (vel linearia)." Figures C, E, F on page 319 of Engler's Pflanzenfamilien show the chromatophores to be mostly elongate, sinuous, sigmoid or U-shaped, forms which we have never observed in our species. According to Thaxter the Florida form has oval or oblong or nearly spherical chloroplasts, which agrees with our species. There can be little doubt that the Bengal species is identical with the species occurring in Florida, and we propose to apply the name of Compsopogon coeruleus to the Bengal alga.

We once more draw attention to the fact that the Bengal plant has been found by us in stagnant ponds, and not in running water.

Quite lately we have discovered a second species of *Compsopogon* in similar localities, a form which, however, is markedly different from the species dealt with above. It will form the subject of a subsequent paper.

BOTANICAL LABORATORY,

University College of Science,

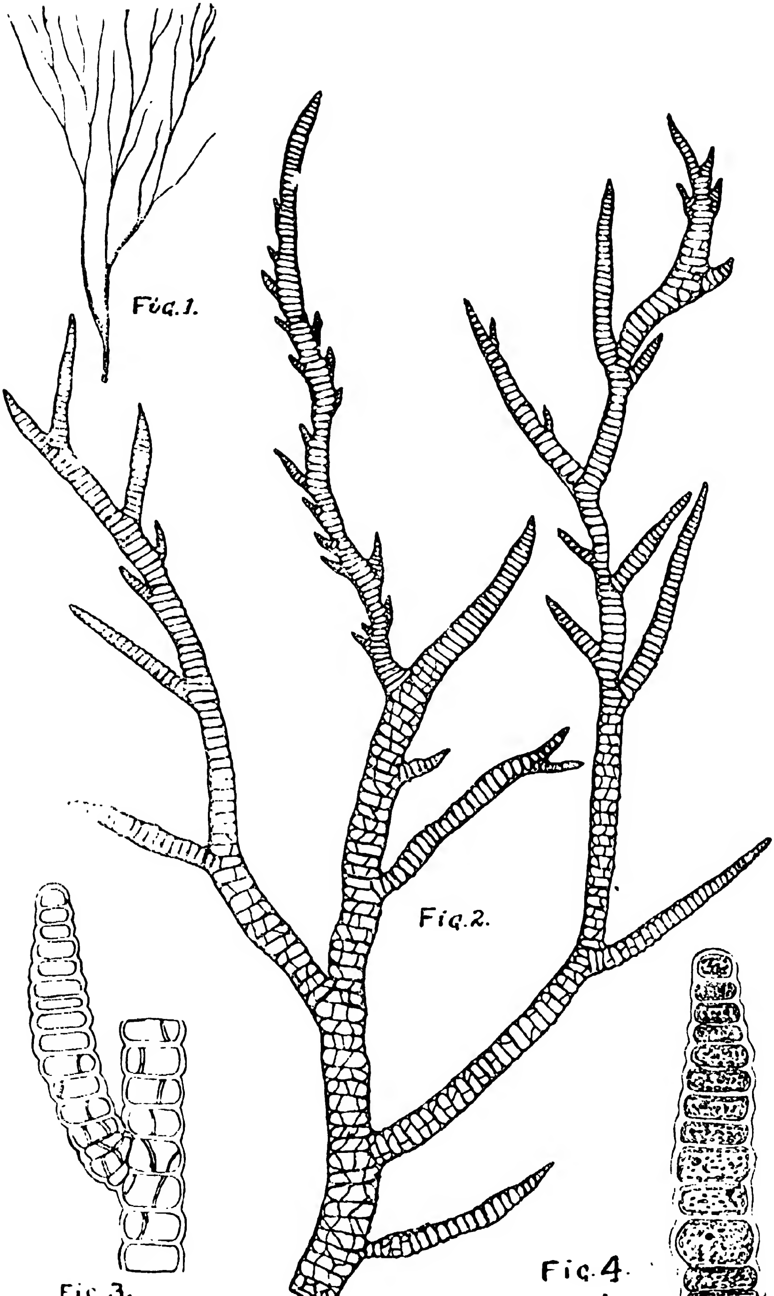
The 25th of March, 1923.

EXPLANATION OF FIGURES.

PLATE VIII.

1. Part of a specimen of Compsopogon sp.; natural size.
2. Upper part of a branch of the second order with branchlets of higher orders; rind-cells developed on the older portions; ×100.
3. Portion of a branchlet of a higher order showing, like the basal cells of the ultimate order, the beginning of the formation of rind-cells by periclinal cell-walls; upper cells of the ultimate branchlet yet undivided; ×250.
4. Upper part of ultimate branchlet, cells with nucleus and chromatophores; ×450.

PLATE VIII



Drawn by K. P. Biswas.

EXPLANATION OF FIGURES.

Plate IX.

5. Young specimens growing on branches of an aquatic grass (*a*); ×100.
6. Two individuals of different ages attached to the culm of an aquatic grass (*a*); rind-cells well developed in the larger specimen; ×200.
7. Rind-cells in face view with contents; ×450.
8. Older branch with a sorus of microsporangia; ×150.
9. Young branchlet with macrosporangia; ×300.

PLATE IX

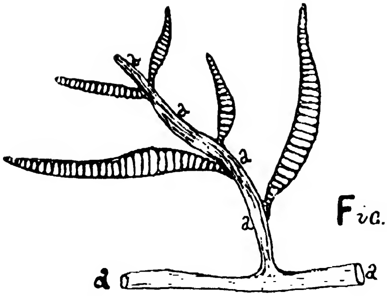


Fig. 5.

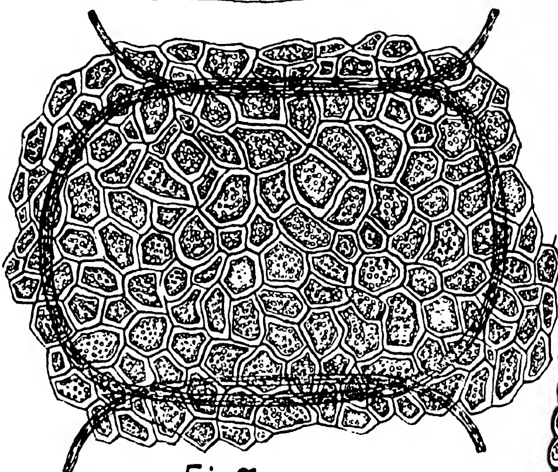


Fig. 7.

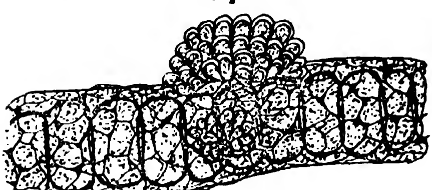


Fig. 8.

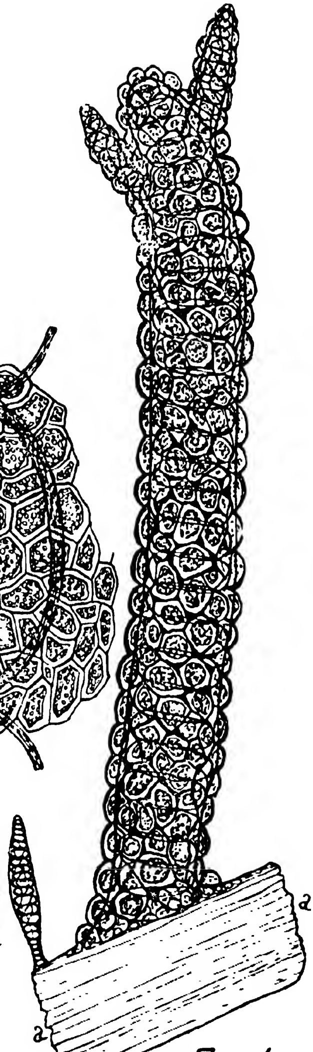


Fig. 6.

Fig. 9.

Drawn by K. P. Biswas.

EXPLANATION OF FIGURES.

Plate X.

10. Young filament with developing microsporangia; × 350.
11. Filament with two sori of microsporangia; × 350.
12. A sorus of microsporangia in surface view; × 450.
13. Group of rind-cells with macrosporangia; × 300.
14. Young filament with macrospores in some of the rind-cells; × 300.
15. Portion of a filament with a sorus of microsporangia; one of the discharged microspores × 500.
16. Microsporangia and a microspore; × 500.
17. Microsporangia; a microspore on the point of being discharged; × about 1000.

18, *a* and *b*. Discharge of macrospore; × 1500.

19. Macrospores; × 1500.
20. Microspores; × 1500.
21. Young filament; discharge of a macrospore; × 400.

22 and 23. Transverse sections through tertiary branchlets of an older filament; × 70.

PLATE X

Fig 10

Fig. 11.

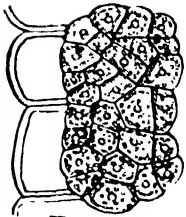


Fig. 12.

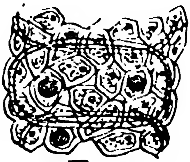


Fig 13

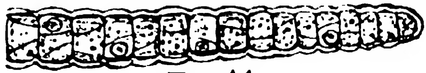


Fig 14

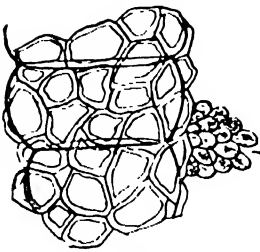


Fig 15

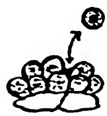


Fig 16

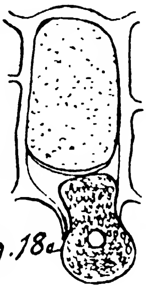


Fig. 18a

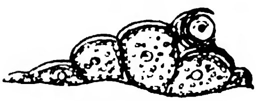


Fig 17

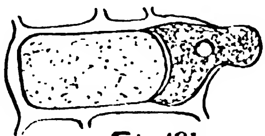


Fig 18b.

Fig 19.

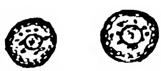


Fig. 20.

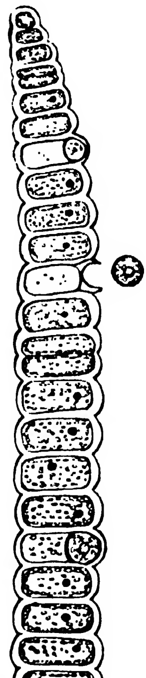


Fig. 21.

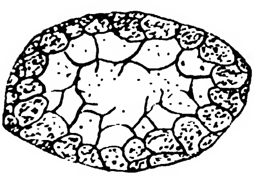


Fig. 22.

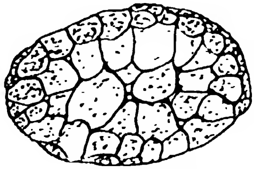


Fig. 23.

Drawn by K. P. Biswas.

EXPLANATION OF FIGURES.

Plate XI.

24. Transverse section through a secondary branch; × 100.

25. Transverse section through the lower part of the main filament of an older specimen; × 50.

26. Transverse section through a very young filament; × 150.

27. Longitudinal section through part of the main filament of an older specimen, of which fig. 25 shows a transverse section; × 50.

28. Transverse section through a young filament with a sorus of microsporangia; × 150.

PLATE X

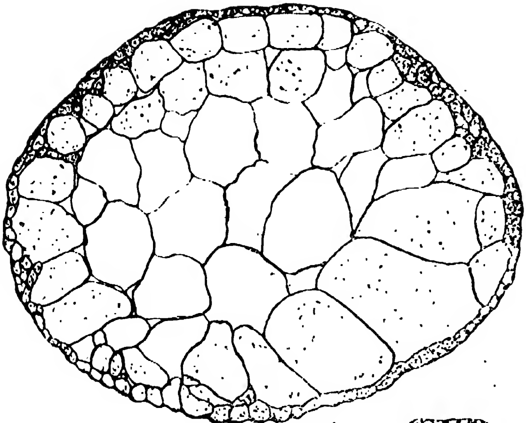


Fig. 24.

Fig. 26.

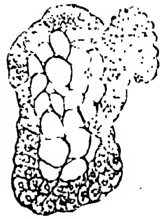


Fig. 28.

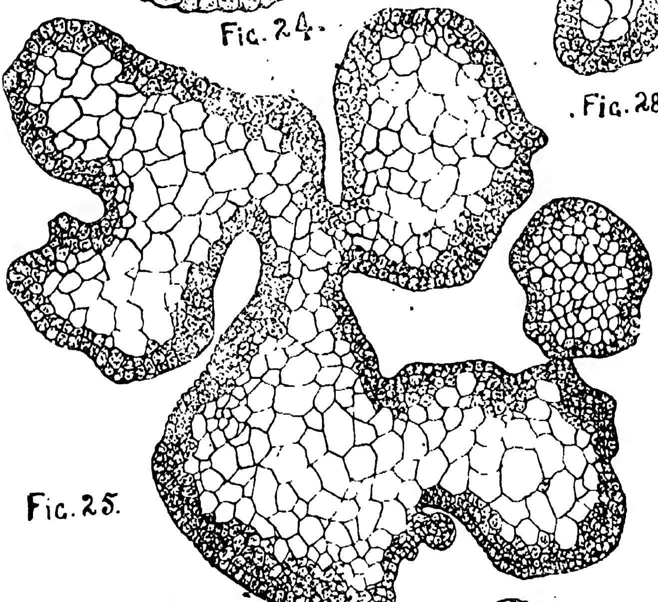


Fig. 25.

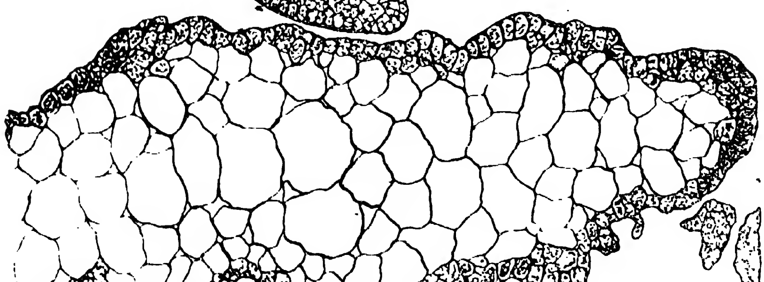


Fig. 27.

COMMENTATIONES PHYTOMORPHOLOGICAE ET PHYTOPHYSIOLOGICAE

II.

EICHHORNIA STUDIES

BY

PAUL BRÜHL, D.Sc., AND ATULCHANDRA DUTTA, M.Sc.

1. MORPHOLOGICAL

The first condition of successfully fighting a formidable enemy is carefully to study his qualities and idiosyncrasies, his powers of offense and defence and the resources at his disposal. Before, therefore, investigating the physiology of the "Water Pest," it will be advisable to deal with the morphological characters, both external and internal, of *Eichhornia speciosa* (=*Eichhornia crassipes*).

SOLMS-LAUBACH, in De Candolle's Monographiae phanerogamarum, Vol. IV, pp. 525-531, describes five species of Eichhornia, all of them indigenous in Tropical America, only one of them being also found in Madagascar and various tropical parts of the African continent. Two of these species—*E. natans* and *E. speciosa*—are normally floating, although the latter may continue to vegetate after being stranded on mud or on layers of vegetable debris, when the water-level is lowered after the cessation of the Monsoon rains.

The external morphological characters of *Eichhornia speciosa* have been so often described in detail, especially since the eradication of the "Water Pest" has become a pressing economical problem in various parts of India and Burma that we may confine our attention to certain features of special interest.

That feature which prominently arrests the attention even of the casual observer is the pseudobulb, which normally develops as part of the leaf-stalk. The factors which induce the formation of these swellings will be dealt with in a subsequent paper, so that for the present we may restrict ourselves to the following remarks.

The pseudobulb is invariably developed in plants which float freely on the surface of the water; such plants continue to bear leaves with bulbous leaf-stalks, even if stranded on the banks of jhils and ponds, after the water evaporates during the dry seasons of the year, when they finally have to continue their existence with their roots buried in mud. We have transferred plants which originally were freely floating into earthen-ware vessels filled with slushy mud; although not growing as vigorously as before, they continued to live, producing short runners ending in rosettes with pseudobulbous leaf-stalks. On the other hand, whole jhils can be seen densely covered with the water hyacinth, the single individuals of which, although still floating, are so crowded together that not even a fraction of a square inch is left for any additional plant to grow. This, as well known, is of common occurrence. In such circumstances the stem, which is surrounded by water, attains a certain length, and the leaf-stalks, which may attain a length of two feet or more, commonly develop no pseudo-bulb at all, tapering from a thicker base to their upper end. When such plants are transferred to vessels filled with water, so that they have more room to grow in, the slender petioles may increase in thickness in their lower part so as to become conspicuously spindle-shaped, therefore showing a decided tendency towards the production of a pseudobulb. What usually appears to happen in the case of these densely crowded individuals is that during the cold weather the older individuals, after throwing out runners, decay, their leaves withering and gradually disappearing, whilst the young rosettes, having now a sufficiency of space to grow in, develop petioles consisting nearly entirely of spherical or pear-shaped pseudobulbs.

The leaf-blades of the free-floating individuals are nearly always pronouncedly kidney-shaped, whilst those of the crowded specimens are larger and more ovate or ovate-oblong. After these leaves have died down and the old individuals have made room for the young offsets, the young leaf-blades are reniform. Something similar happens when individuals with thinner, tapering petioles are transferred to large earthen-ware vessels (gamlas) containing Sachs' culture solution or a solution containing 1 gram of potassium nitrate, 0·25 gram of dihydrated calcium sulphate, 0·25 gram of magnesium sulphate, 0·5 gram of hydrogen potassium phosphate, and a few drops of ferric chloride. No appreciable difference was noticed with plants

growing in gamlas filled with ordinary clean pond water and vessels filled with the culture solution referred to above. The rosettes which were cultivated in moist mud continued to produce kidney-shaped leaf-blades and petioles with small globular pseudo-bulbs; but no tendency to develop thin spindle-shaped leaf-stalks was observed.

Mr. Jivanna Rao, in his paper on the formation of leaf-bladders in Eichhornia speciosa (Journal of Indian Botany, vol. I, p. 219 and ff, 1920) distinguishes four types: "(1) all leaves with bladders; (2) no leaves with bladders, (3) outer leaves bladdered, inner bladderless; (4) inner leaves bladdered, outer bladderless." Type (1) is the typical form of free-floating plants; type (2) is commonly found in the rainy season in jhils densely overgrown with the water hyacinth; type (4) is evidently formed when type (2) happens to get into positions where the crowding is relieved in some way or other, and is found also more commonly in the cold season; type (3) is probably developed when, during the rainy season, the originally bladdered form becomes more and more crowded, which state of things would result in the stem elongating and the stalks of the younger leaves becoming longer and more slender. Without going into the real causes of these differences at the prseent moment, the elongation of stem and petioles may be compared to what happens when various species, such as *Digitaria sanguinalis, Solanum nigrum, Acalypha indica, Phyllanthus Niruri* and others, attain considerable lengths of stem (and branehes) when growing in dense thickets formed by taller species. It is noticeable that in the case of jhils overgrown with the water hyacinth during the rainy season the plants growing near the edge of the plant mass are much shorter and bear bulbous leaf-stalks, whilst the main mass further in consists of plants with longer stems, more slender petioles and more ovate leaf-blades. These features give the whole plant mass a convex transverse section, whilst during the cold season the transverse section of the plant mass is more uniform, and the mass is less dense. These facts seem to indicate that if any mechanical method of clearance is adopted —and this, it appears to us, is the only feasable method of getting rid of the pest—this clearing, quite apart from other reasons, should be done in the cold season.

In the free floating form the internodes of the stem remain exceedingly short, giving thus rise to the formation of rosettes of leaves; the densely crowded, although floating plants possess, when

fully developed, a short and thick cylindrical stem ; the latter consists of the stem proper and a peripheral continuous pseudo-cortex consisting of the very bases of the petioles, as shown on plate I, fig. I. This feature will be dealt with in a subsequent paper.

The primary root, as is well known, soon dies, its functions being transferred to adventitious roots, which take their origin in the periderm of the stem, invariably break through the leaf-sheaths in a more or less horizontal direction and together form ultimately a dense bunch hanging down into the water. These adventitious numerous short secondary rootlets stand nearly at right angles to the parent primary rootlet. No roothairs are observable in floating plants, but they develop on plants growing in wet mud. The adventitions roots and rootlets remain colourless, when the plants grow in dense masses or in mud or vegetable debris ; but when free-floating and particularly in the drier seasons of the year, the bunches of adventitious roots are coloured purplish blue or even assume a dark indigo-coloured hue, due to the copious formation of anthocyanin in the stronger and more continuous illumination from sun and sky after the close of the monsoon. (See also the excellent figure on plate III opposite page 148 of Principal G. C. Bose's Manual of Indian Botany). The colouring matter is not only dissolved in the cell-sap, but tinges also the cell-walls of roots and rootlets and diffuses into the walls and contents of two or three layers of cells of the cortical tissue of the main adventitious roots. Red colouring matters develop often on the parts of the pseudo-bulbs and in the runners immersed in water.[1])

Goebel, in Pflanzenbiologische Schilderungen, part I, p. 6, suggests that a purpose secondarily served by the pseudobulbous leaf-stalks may be the righting of the floating rosettes, after they have been turned over by wind or waves. This, at first sight, seems to be verified by experiment ; for when a rosette is thrown upside down on a surface of water, the plant quickly turns back into its normal position ; if, however, the bunch of adventitious roots is clipped off with a pair of scissors and the experiment repeated, the plant is unable to right itself and continues to float upside down. It may be

[1] *Various rather hysterical statements have been made by certain newspaper correspondents. Thus it has been stated that the colouring matter escapes into the surrounding water colouring the water blue and poisoning it. This has no foundation in fact. As a curiosity it may be mentioned that certain people in Eastern Bengal mix the flowers of the Water Hyacinth with flour and fry the mixture.*

mentioned here that if a plant bearing only leaves with non-bulbous petioles is thrown on a free surface of water, it is unable to float in the upright position. One of the results of the existence of pseudo-bulbous petioles in freely floating individuals is their greater resistance to bending as compared with the resistance offered by an equal length of the slender leaf-stalks, as can easily be verified by a simple experiment. It must, however, be noticed that the cuticle of the bladdered petioles is probably always thinner than that of the bladderless ones. In any case, the greater stiffness due to the presence of the pseudobulb makes the bladdered stalk a more efficient mast to which the comparatively broad leaf-blade is attached as a kind of sail in the freely floating form.

By far the most common form of reproduction is that by means of runners. In the axil of a leaf arises a bud, which at first develops into a miniature sessile rosette, which, after having attained a certain size, is carried by the development of a stalk-like structure to a certain distance from the parent plant; the rosette rapidly grows in size, and, the stalk being brittle, ultimately may become separated and embarks on a separate career, may be, at some distance. The daughter rosettes, in their turn, produce new rosettes in all directions, and this process may be continued indefinitely, until, especially in stagnant pools, large colonies are formed, often constituting an extensive and closely packed network.

Statements have been made that *any* part of a plant separated from the main stock can reproduce the species. According to our own observations this is hardly correct. Leaves cut off slightly above their base and thrown into water soon decay; but if the cut is made so as to include a small portion of the stem, a bunch of adventitious rootlets soon arises from the stem part and a new plant originates from the fragment. Further, if the stem is cut across in two places and the cut portion split longitudinally, buds arise from the cut surface and new plants are formed. As a matter of fact, any part of the *stem* will regenerate the plant, when thrown into water.

The intrapetiolar stipules will form the subject of a subsequent paper.

2. ANATOMICAL

The Tegumentary System.

The epidermis consists of a single layer of cells. These cells are somewhat irregularly oblong on both sides of the leaf-blade as well as on elongated, spindle-shaped petioles, whilst those of the pseudobulbs are more isodiametric and often nearly regularly pentagonal, hexagonal or heptagonal. The outer wall, including the cuticle of the epidermal cells of the leaf-blade, is about 2·2 micra in the central part of the leaf, but somewhat thinner towards the margin, while on bladderless leaves the outer walls of the cells of the fully developed petioles is 4 to 4·5 micra in thickness, that of the mature pseudobulbs of bladdered leaves being 2 to 2·4 micra. The cells of the tegumentary system contain no chloroplasts.

The Conducting System.

This system consists of thin fibro-vascular bundles of the ordinary monocotyledonous type. In very young leaves, before they have unfolded, there is seen a minute subcylindrical projection, the base of which is situated at the lower surface of the apex of the leaf-blade. This projection consists of an envelop of loosely packed cells surrounding a core of a few cells, several spiral vessels being seen imbedded in the core. In slightly older leaves a narrow strip at the lower apical part of the leaf-apex assumes a brown colour, the projection mentioned above falls off and its place is taken by a slight depression. The ordinary stomata of this region are transformed into water stomata, losing their chloroplasts at the same time. The number of water stomata increases with the age of the leaf. In longitudinal sections it is seen that the vessels closely approach the apex giving place to a number of tracheids with spiral thickenings. As already stated by Max von Minden in his "Beiträge zur anatomischen und physiologischen Kenntnis Wassersecernierenden Organe" (*Bibliotheca Botanica, vol. IX*), Eichhornia is remarkable for the copious secretion of water. It may be added that the size of the water stomata varies from 42 μ × 31·5 μ to 33 μ × 33 μ, whilst the projection referred to was, in one case, measured to be 265 μ in length, the thickness being about the same.

The Ventilating System.

The secondary rootlets possess no intercellular spaces ; the primary rootlets, on the other hand, have a well developed aërenchyma situated within the vertical region and consisting of short-prismatic air-spaces. In the stem the ventilating system is formed of intercommunicating narrow intercellular spaces; only comparatively rarely well-developed air-canals, interrupted by transverse diaphragms, are observable in the central portion of the stems.

The petioles, both the bladdered as well as the bladderless ones, contain air-canals, which, as a matter of fact, are much more numerous in the former than in the latter and which in both cases are widest in the central portion and narrowest in the peripheral regions. In a particular case, the diameter of the central air-chambers varied between 5 and 2 millimeters ; along a radius the consecutive measurements were 3mm., 2·5mm., 2mm., 1mm., 750μ., 600μ, whilst the peripheral ones were 225μ, 150μ, 105μ, 90μ in diameter.

These air canals are bounded each by layers of single thin-walled cells, which in cross-section are subquadrate or oblong with rounded corners ; a longitudinal section shows the boundary cells to be arranged in one or more tiers of mostly oblong cells filled with watery usually non-granular contents separated by a few layers of mostly transversely oblong or irregularly shaped smaller cells containing numerous starch grains. The single air-canals are partitioned off into compartments by transverse diaphragms. These diaphragms consist of a single layer of cells, which at and near the margin are mostly oblong in shape, whilst the inner ones are practically isodiametrical, the cross-dimensions of the former being from 56 × 45 to 130 × 49μ, those of the latter being 18—36μ. In the bladderless leaves intercellular spaces are seen to be present between all the cells of the diaphragms, except between a few peripheral cells of the diaphragms of the central canals ; the intercellular spaces, where they occur, are 5-9μ in width ; in the bladdered leaf-stalks, on the other hand, the intercellular spaces of the diaphragms diminish in width from the circumference of the stalk towards the centre, where they are usually entirely absent. The dimensions of these intercellular spaces vary between 2 and 4μ. It is noticeable that the intercellular spaces become narrower and narrower and finally disappear as we pass from the circumference of the petiole towards its centre,

as well as from the periphery of the individual diaphragm towards its centre.

By the aid of Edinger's projection apparatus we have been able to count the number of air-canals over various cross-sections of the leaf-stalk. Thus in a set of sections through a bladderless petiole we found 224 canals at the apex, 359 a few centimeters below the apex, 463 at the middle part, and 490 at the thicker basal end. In bladdered leaf-stalks the number of air-canals varies greatly with the age and size of the pseudobulbs. Thus in the bladdered leaf-stalk of a very young leaf we counted per cross-section 112 canals in the upper part of the bladder, 186 in the middle, and 166 at the base; the corresponding numbers in the case of a fully grown leaf of moderate size were 254, 407 and 273, whilst the numbers in the case of a larger, fully grown leaf were 715, 842 and 648. In the latter case, therefore, the number increased from the base to the middle by nearly eighteen per cent., whilst they diminish upwards by nearly thirty per cent.

The increase in the number of air-canals from the base towards the middle of the pseudobulbs is due partly to the independent formation of new air-canals, partly to the bifurcation of already existing ones, whilst the reduction from the middle upwards has apparently its cause in the abrupt ending of certain canals in the upper and thinner end of the petiole.

It was interesting to gain some idea about the volume of gases present in a pseudobulb; in the case of a bulb arbitrarily chosen, the bulb, which had a volume of 19 cc. contained more than 16 cc. of gases, so that in this case the air occupied nearly 85 per cent. of the total volume of the bladder.

As the boundary cells of the air-canals contain chiefly water, we determined the percentage loss of water on drying at 100°C, and found it to be 94 per cent. in the case of a bladder and 90 per cent. in the case of a bladderless petiole; similar results were obtained with entire leaves, blade plus petiole.

The air-canals extend into the leaf-blade, accompanying the fibrovascular bundles and spreading out with the latter, whilst as they approach the apex of the leaf-blade their width diminishes and finally they disappear, before they reach the apex.

Starch grains occur in considerable numbers in the cells of the transverse diaphragms of the bladderless petioles, but in the

corresponding cells of the pseudobulbs they are much scarcer and usually of smaller size, or they are entirely absent, particularly in the central cells of the diaphragms. It may also be at once mentioned that starch grains occur in large numbers in the cells of the stem as well as in those of the continuous rind formed by the leaf-bases. Starch is also found in the inner cortical cells of the primary rootlets. The diameter of the starch grains varies from 7 μ downwards.

The stomata will be dealt with in a future paper.

The Photosynthetic System.

The chloroplasts are contained in a few layers of cells below the epidermis on all sides of the bladderless leaf-stalk, whilst in the pseudobulbs the photosynthetic system is confined to the upper part more exposed to sun-light. In the leaf-blade chloroplasts occur in both the palisade and the spongy tissues. Starch grains are plentiful in the cells of the photosynthetic system.

Finally, we may refer to the Rhaphides. These occur either singly, traversing the longitudinal walls of the air-canals, often projecting into adjacent canals at either end and varying in length in various cases measured by us between 150 and 250 μ and in width between 7 and 11μ, or they occur in bundles contained in particular cells associated with transverse diaphragms, where they vary in length between about 50 and 80μ. These bundles are especially numerous in the stem and also in the cells near the apex of very young leaves, before the latter assume a green colour.

BOTANICAL LABORATORY,
CALCUTTA,
UNIVERSITY COLLEGE OF SCIENCE,
BALIGANJ,
The 15th of March, 1923.

EXPLANATION OF FIGURES.

Plate XII.

Fig. 1. Transverse section of a bulbless petiole, showing epidermis with two stomata, part of the photosynthetic system, two fibro-vascular bundles, and portion of the aërenchyma, one of the air-chambers with transverse septum, × 113.

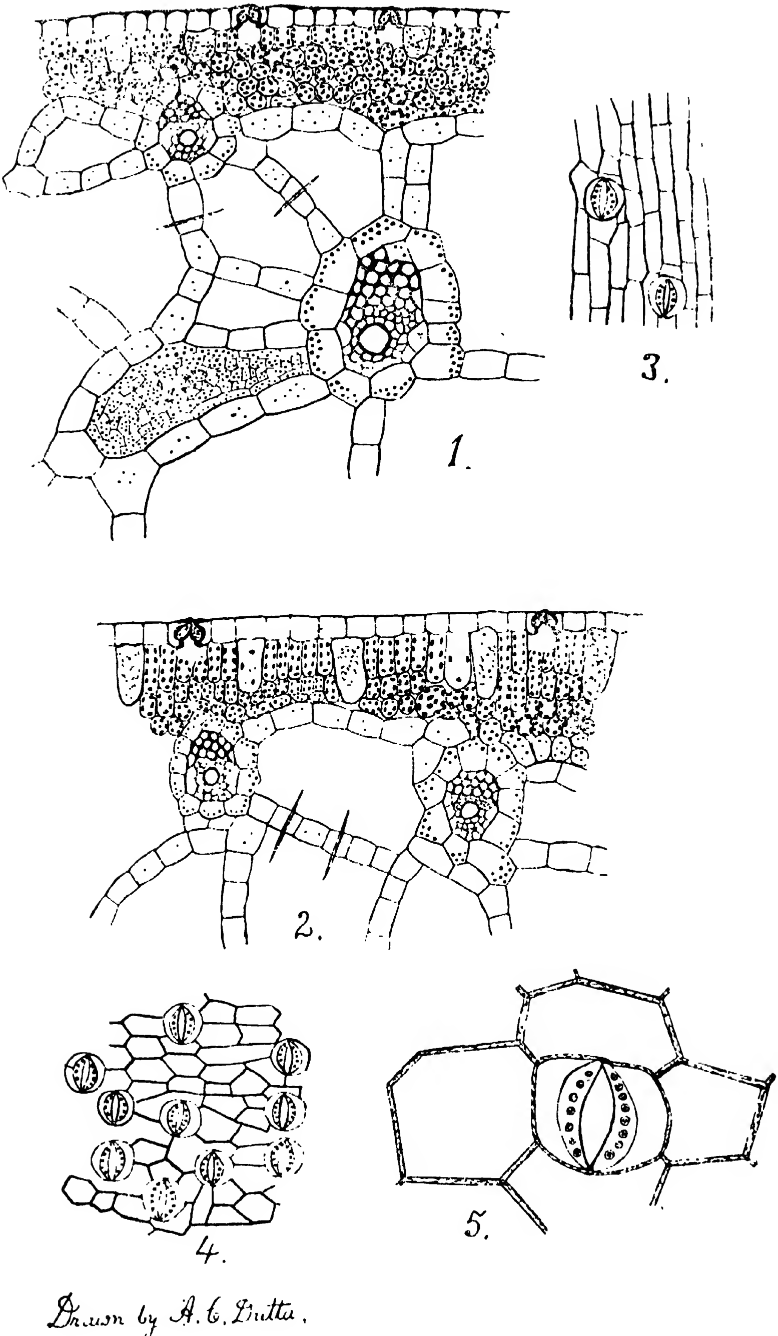
Fig. 2. Part of transverse section through a pseudobulbous petiole, × 113.

Fig. 3. Surface view of epidermis of a pseudobulb, × 113.

Fig. 4. Surface view of epidermis of a bulbless petiole, × 113.

Fig. 5. Stoma, × 500.

PLATE XII



Drawn by A. C. Dutta.

EXPLANATION OF FIGURES.

PLATE XIII.

Fig. 6. Partition wall of an air-chamber in longitudinal section, $\times 113$.

Fig. 7. Surface view of epidermis of leaf-blade, $\times 113$.

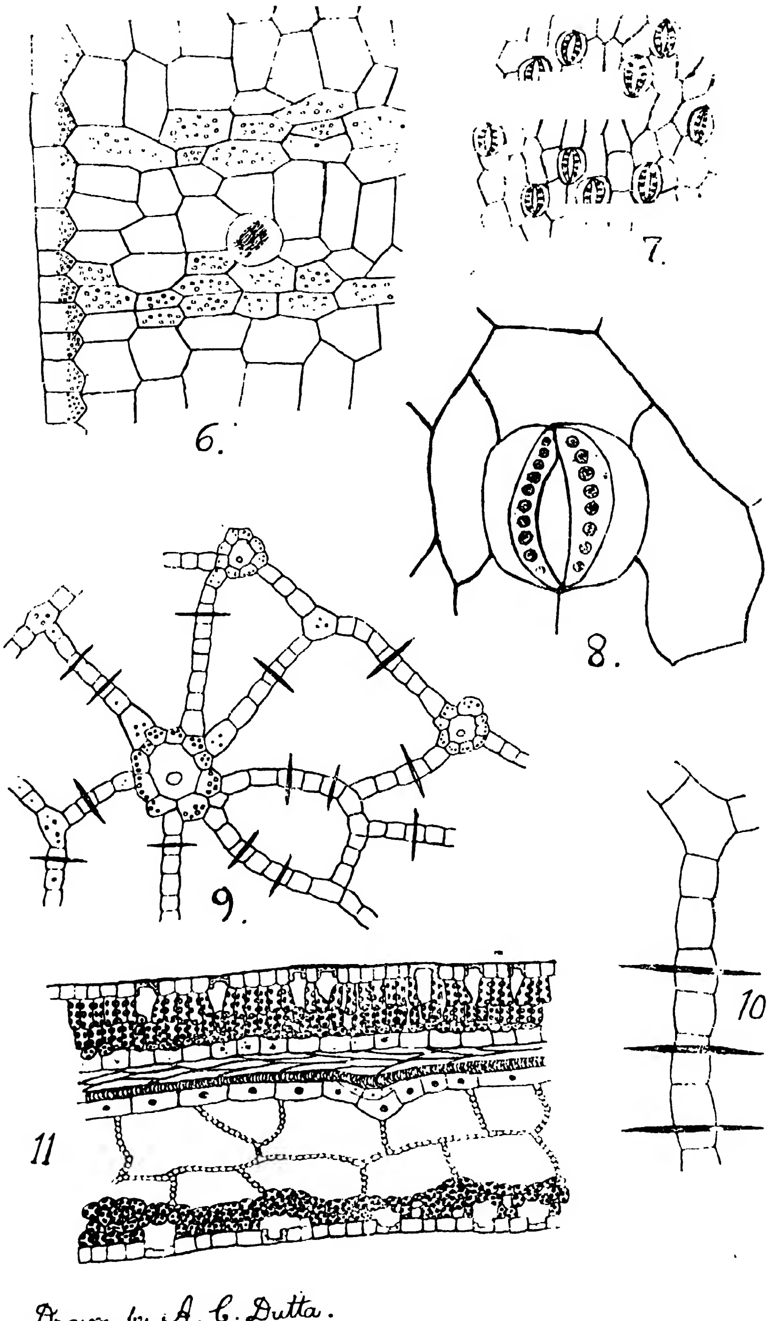
Fig. 8. Surface view of a stoma from the leaf-blade, $\times 500$.

Fig. 9. Transverse section through the central portion of a pseudobulb, $\times 18$.

Fig. 10. Partition-wall of air-canal in transverse section, $\times 113$.

Fig. 11. Longitudinal section of leaf-blade, showing upper and lower epidermis, palisade and spongy tissues, numerous air-chambers, and a fibro-vascular bundle.

PLATE XIII



Drawn by A. C. Dutta.

EXPLANATION OF FIGURES.

PLATE XIV.

Fig. 12. Transverse section through leaf-blade, showing upper and lower epidermis, photosynthetic and conducting system and aërenchyma, $\times 58$.

Fig. 13. Stoma in transverse section, $\times 500$.

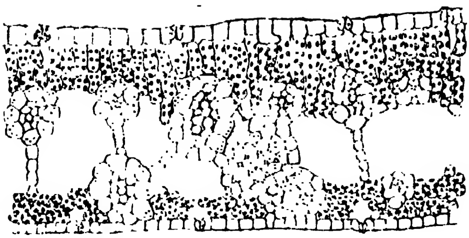
Fig. 14. Peripheral transverse diaphragm of bulbless petiole, showing comparatively large intercellular spaces and cells with starch grains, $\times 113$.

Fig. 15. Transverse diaphragm of air-canal from the more central portion of a bulbless petiole; no intercellular spaces between the peripheral cells; starch grains diminish in size from the periphery to the centre of the diaphragm, $\times 113$.

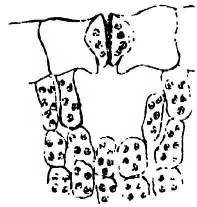
Fig. 16. Diaphragm of an air-canal situated near the circumferential portion of a pseudobulb; cells near the periphery of the diaphragm (on the right) larger, without intercellular spaces and larger starch grains.

Fig. 17. Central portion of a transverse diaphragm from the inner part of a pseudobulb, without intercellular spaces and no starch grains, $\times 113$.

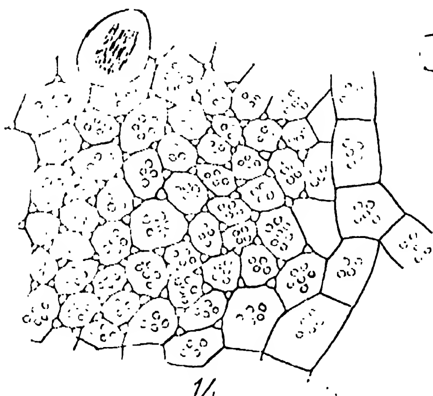
PLATE XIV



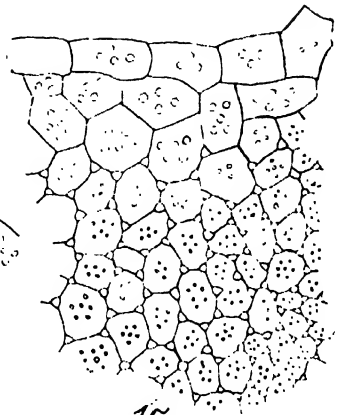
12.



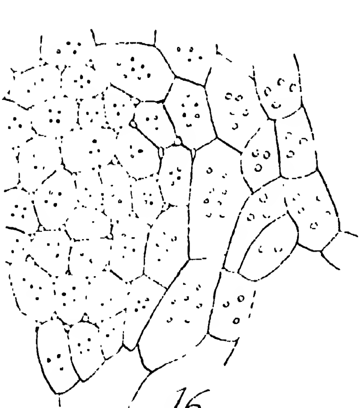
13.



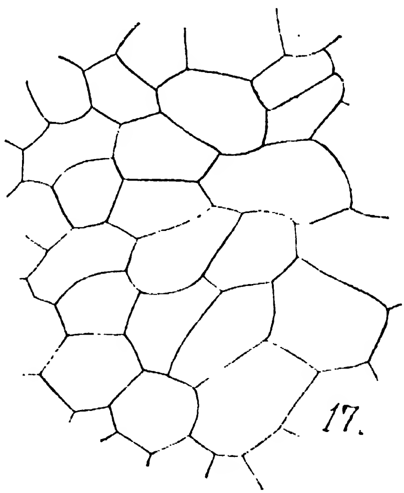
14.



15.



16



17.

Drawn by A. C. Dutta.

EXPLANATION OF FIGURES.

Plate XV.

Fig. 18. Peripheral part of the diaphragm shown in fig. 17 on plate XIV; no intercellular spaces and a few small starch grains.

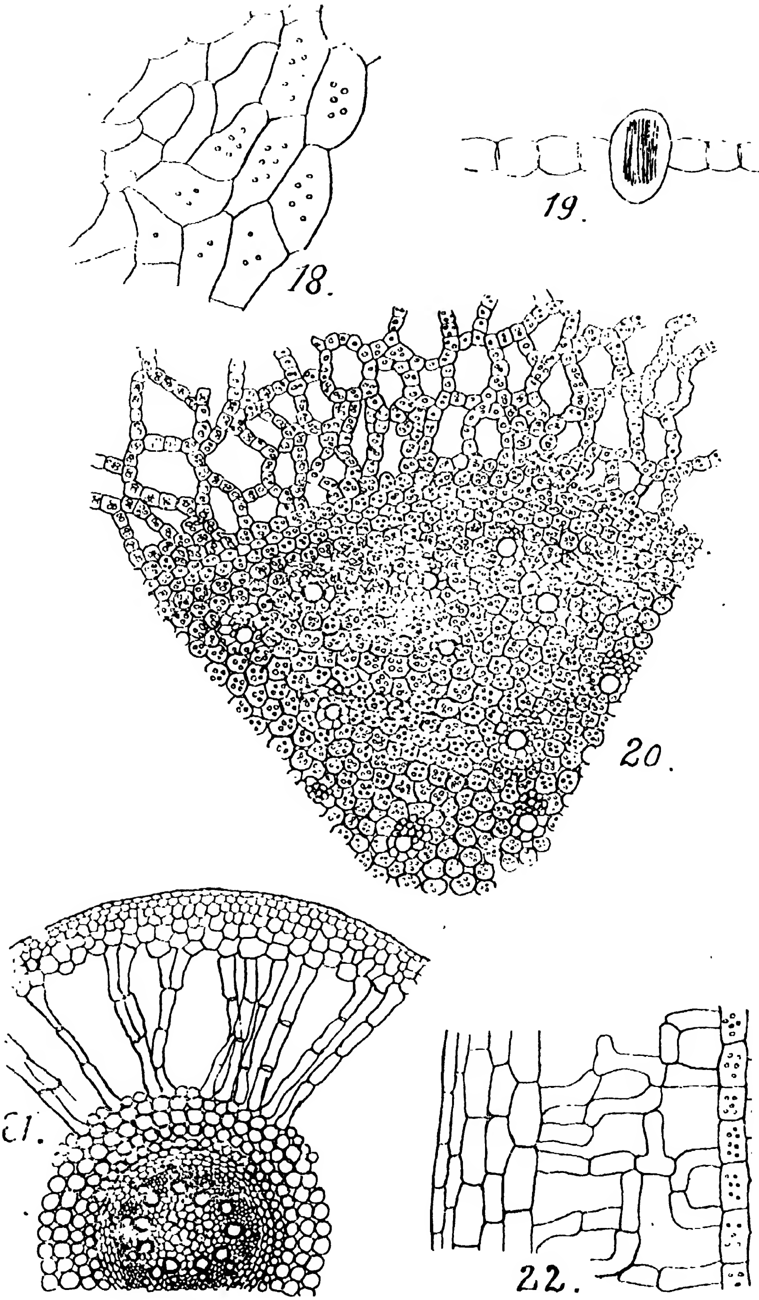
Fig. 19. Some of the boundary cells of an air-chamber; one of the cells much larger and containing a bundle of rhaphides.

Fig. 20. Transverse section of stem; inner portion (below) with numerous fibro-vascular bundles and cells of the ground-tissue full of starch grains; rind portion (above) formed of leaf-bases traversed by air-canals. From the more elongated stem of a plant with long-petioled bulbless leaves, $\times 58$.

Fig. 21. Transverse section through a primary adventitious rootlet, showing large air-spaces, $\times 58$.

Fig. 22. Longitudinal section through a primary rootlet; epidermis on the left, $\times 113$.

PLATE XV



Drawn by A. C. Dutta

EXPLANATION OF FIGURES.

Plate XVI.

Fig. 23. Apical protuberance of a very young leaf; five vessels converge towards the tip of the protuberance, × 113.

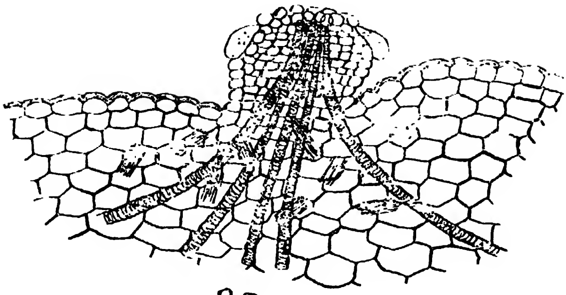
Fig. 24. Longitudinal section through leaf-apex after the dropping of the protuberance; a number of tracheids are seen, × 180.

Fig. 25. Part of lower surface of leaf-apex with water stomata, × 180.

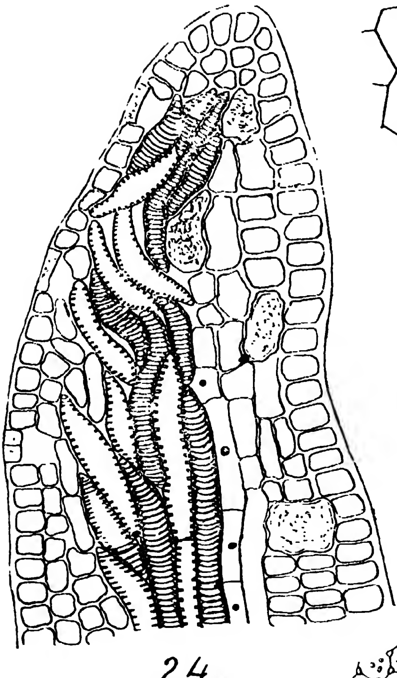
Fig. 26. A single water stoma, × 500.

Fig. 27. Portion of a transverse diaphragm of an air-canal of a leaf-blade, × 113.

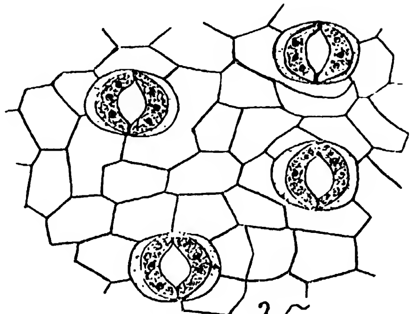
PLATE XVI



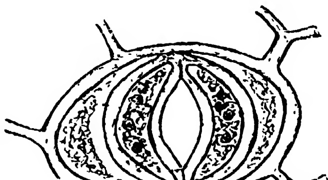
23.



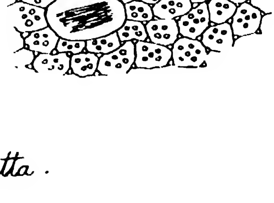
24.



25.



26.



27.

Drawn by A. C. Dutta.

COMMENTATIONES PHYTOMORPHOLOGICAE ET PHYTOPHYSIOLOGICAE[1]

BY

PAUL BRÜHL, D.Sc.

I

ON VITAMINES

The present paper is in substance part of a lecture, delivered before the Botanical Section of the Science Congress held at Lucknow in January last, on "New Avenues of Botanical Research."

The nature and origin of life is undoubtedly the central problem of biology; and any new avenue which holds out a promise of leading in a direction towards the solution of that all-important problem is well worth attracting the attention of biologists, even although not every one interested in biological studies may have time, opportunities, or inclination to proceed along even one of those avenues. When I mention new avenues of research, I do not refer to avenues which have been planted only yesterday. The atomic theory, for instance, was conceived nearly three thousand years ago, and yet it has advanced with giant strides only during the last twenty-five or thirty years, and the epoch-making discoveries regarding the structure of the atom, the result of painstaking work and the application of analytical powers of the highest order, are of very recent date indeed.

Among the comparatively new lines of research which finally must deeply, perhaps fundamentally, affect the progress of biological science I may draw attention only to the following: Intensive application of physio-chemical methods to the study of the vital processes which go on uninterruptedly in the living cell; the utilisation, in the investigation of physiological prob'ems, of the advances

[1] It is proposed to publish under this heading the results of morphological, anatomical and physiological researches carried out in the botanical laboratory of the College of Science, University of Calcutta.

steadily made in the province of Kolloid Chemistry; the application of the recent developments of the Atomic Theory, including radio-activity, the doctrine of isotropes, the structure of the atom and similar subjects, to the solution of problems which carry the physiologist into ultra-molecular regions; ultra-microscopic researches into structures and events beyond the reach of the microscope.

Not entering, however, at the present instance, on any of these avenues of research, I propose to confine myself to a subject which up to the present time has attracted comparatively little attention among botanists, and that is the subject of Vitamines.

It were the investigations into the ætiology of three dreaded diseases—Scurvy, Beriberi, and Rhachitis—which led to the discovery of a class of substances, found in vegetable tissues, to which various names have been given, among which the shortest is that introduced into science by Casimir Funk in the first edition, published in 1913, of his book entitled "Die Vitamine. Ihre Bedeutung für die Physiologie and Pathologie." The second edition of the work appeared early in 1922.

At the present moment it is safe to assume the existence of at least four types of vitamines: the anti-rhachitic or A vitamine, the anti-beriberi or B vitamine, the anti-scorbutic or C vitamine, and a fourth or D vitamine which stimulates the growth of organisms. Even this does not exhaust the number of probably distinct vitamines, but of these the existence of the four just mentioned has been clearly established.

The majority of the vitamine investigations has been carried out by experiments on animals, such as rats, dogs and pigeons, or by observations in hospitals and lunatic asylums; but there are strong hopes that investigations on the part played by vitamines in plant life will lead to discoveries which will enable biologists to substitute shorter-time experiments on members of the vegetable kingdom for the somewhat lengthy experiments which have been carried out on animals, chiefly by members of the medical profession. In this direction experiments with yeast have yielded promising results. Observations made by PASTEUR as early as 1871 led WILDIERS to experiment with culture solutions containing salts of ammonium as the only source of nitrogen and to which, besides some living yeast cells, a small quantity of a sterile extract of yeast was added. WILDIERS concluded that, besides *known* substances there must be

present in the yeast extract some substance, then yet unknown, which he called *Bios* and which he supposed to produce a growth impulse on the colony of living cells. That substance was found to be soluble in water and in eighty per cent. of alcohol, but insoluble in ether, and to be dialysable. The properties of this "bios" were therefore similar to what now is known as the B vitamine. It is highly probable that yeast cells are particularly suitable for testing for vitamines, especially those belonging to the B type. One of the most important contributions to the subject is J. R. W. Williams' paper on "The vitamine requirements of yeast. A simple biological test for vitamine" in the journal of Biological Chemistry, XXXVIII, p. 765 (1919). One of the important results of investigations was the discovery that B vitamine is readily absorbed by Fuller's arth. FUNK describes a convenient method of determining the activity of the vitamine which accelerates the growth and multiplication of yeast cells. Further experiments have shown that this vitamine is not identical with the anti-beriberi vitamine and is therefore called D vitamine by FUNK. It is this vitamine which will probably prove to be the most promising of its class in experiments with lower organisms.

Regarding the anti-rhachitic or A vitamine it has to be stated that the only method yet discovered to prove its presence in any product consists in experiments with rats. From experiments of this nature it follows that, besides in codliver oil, it also occurs in seedlings and in green leaves.

A great deal remains yet to be learned about vitamine C, the anti-scorbutic vitamine. It is certain to occur in vegetables, oranges, sweet lemons, grapes, raspberries, strawberries and pine-apples. An interesting observation was made by ALICE SMITH, namely that, although C vitamine occurs in the lime prepared from Southern European limes, it is absent from the lime juice manufactured from West Indian sour limes. Mr. Howard informed me that this difference is due to the two plants from the fruits of which the Mediterranean and the West Indian lime-juices are prepared are not co-specific (or co-varietal ?). This raises an interesting question to which I shall refer later on.

The B vitamine appears to be the most widely distributed in the vegetable kingdom. It probably occurs in the grains of all Graminaceae, in the seeds of Papilionaceae, in buckwheat, cauliflower,

potatoes, carrots, tomatoes, onions, spinach, oranges, lemons, grapes, bananas, walnuts, cocoanuts, chestnuts, hazel-nuts and almonds.

The A vitamine has been proved to be present in maize, oats, alfalfa, timothy grass, cauliflower, spinach, sweet potatoes, salad, tomatoes and carrots, and in small quantities or doubtfully in other vegetable products.

The C vitamine has been found in the germinating seeds and in the seedlings of barley, in peas, beans, lentils, in cauliflower, potatoes, raw tomatoes, sun-dried onions; particularly rich in vitamines of the C type are oranges, citrons, somewhat less so grape juice, apples, bananas, tamarind, mango. It may be noticed that cabbage, raw or rapidly dried, but not boiled, fresh carrots, raw tomatoes and bananas contain A, B and C vitamines, and that long continued drying or steaming or boiling appears either to diminish or entirely destroy one or the other of the vitamines.

Of particular interest to the botanists are the results of the experiments carried out by BOTTOMLEY, ROSENHEIM, and MISS MOCKERIDGE. BOTTOMLEY experimented at first with young plants of the tomato, buckwheat, radishes and oats, which were grown in a nutritive solution to which was added an extract indirectly obtained from the peat *auximone*. In later experiments BOTTOMLEY used *Lemna minor*, *Salvinia natans*, and a species of *Azolla* as test plants. ROSENHEIM found that very small quantities of the extract from fermented peat added to the soil of pot plants of a species of *Primula* caused a remarkable increase in the vigour of growth of the test plant. MISS MOCKERIDGE investigated the effect on the growth of *Lemna major* of extracts from putrifying leaves and of bacterized peat. The influence on the growth of test plants exercised by rotting leaves was found by her to be about twice that, and that due to bacterized peat about five times that of an ordinary culture solution. The difference of effect on the growth of plants due to fresh and rotten dung is probably at least partly ascribable to the production of vitamines by bacteria.

A large amount of work has yet to be done on the subject. Of prime importance is the elucidation of the chemistry of the different types of vitamines; this is chiefly the business of the biochemist. A further most important problem is the extent to which soil-bacteria and soil algæ are responsible for the production of vitamines. To this chapter belongs evidently the subject of the symbiosis of bacteria

with seeds and the influence which the latter exercise on the process of germination of the former, and perhaps also the symbiosis of certain algæ such as *Anabaena* with Azolla and some phanerogams. We have further the problem whether green leaves have the power of manufacturing one or several vitamines; why closely related varieties or species differ from each other by the presence or absence, as the case may be, of a certain type of vitamine; why some species are able to produce at least three types of vitamines, whilst others may be unable to manufacture even one of them; what is the significance of certain vitamines occurring in fruits such as those mentioned above. Considering the truly wonderful rapidity with which in a few weeks or even days *Clathrocystis aeruginosa*, various species of *Lemna*, *Azolla*, *Trapa*, perhaps also *Eichhornia speciosa*, *Pistia Stratiotes*, *Salvinia natans* spread over the surface of pools, ponds and jhils, it would appear to be worth while investigating how far vitamines are concerned in the process.

Finally it may be pointed out that a detailed investigation into the occurrence of the vitamines in Indian fruits, seeds and vegetables other than those already mentioned would be certain to lead to interesting results, and that the nutritive value of Indian foodstuffs is a subject which requires being thoroughly overhauled in the light of the results of recent vitamine researches.

BOTANICAL LABORATORY,
UNIVERSITY COLLEGE OF SCIENCE,
Baliganj, Calcutta, the 12th of March, 1923.

GEOLOGY

INDIAN PRE-HISTORY.

BY

HEM CHANDRA DAS-GUPTA, M.A., F.G.S.

Introduction.

I was agreeably surprised at the kind invitation of Dr. Simonsen, our worthy Secretary, to deliver a public lecture in connection with the present session of the Indian Science Congress and, as soon as I selected the present topic as the subject of my lecture, the memory of one man whom I had the privilege of meeting a few years ago on the top of the Shevaroy hills arose in my mind. Among the many illustrious geologists, including the President of the present session of the Congress, to whose indefatigable labour we owe our knowledge of the geology of this part of the Indian Peninsula, the late R. B. Foote's was a very prominent figure. But I presume that the late Mr. Foote who, evidently after his retirement from active service, adopted this Presidency as his home, is known to the wider public more as a student of the pre-historic geology of India than as a student of its Archæan time. Mr. Foote joined the Geological Survey of India in 1858. After retirement from the Government service in 1891 he joined the States of Baroda and Mysore as a geologist and on the completion of his work in these States he devoted his well-earned leisure to the study of the Indian pre-historic antiquities till his death which took place in 1912.[1] According to a statement published by the late Dr. Ball[2] the credit of the first discovery of a pre-historic stone implement *in situ* belongs to the late Mr. Foote, the description of which was one of his earliest works while his last contribution dealing with the pre-historic and protohistoric antiquities of his own collection and preserved in the Madras Government Museum was published posthumously by the same Government in 1914.

[1] Q. J. G. S., Vol. 69, pp. lxv-lxvi, 1913.

[2] Proc. Roy. Irish Acad., Ser. II, Vol. I, p. 403, 1879. (Polite literature and antiquities.)

Methods of Study.

Anthropology finds a place now in the curricula of some Indian Universities, but it appears that in all cases the subject is not pursued with the proper scientific bent. Like all other branches of scientific study, any conclusion that is to be drawn in the domain of anthropology must be based on well-established facts. Thus, while no evidence has been found as yet on which the existence of the Tertiary man can be unquestionably placed, it is extremely surprising to see an author [1] stating that the human culture reached such a high state of perfection in the upper Tertiary times as rendered the composition of the Vedas possible. The absurdity of this statement is so clear that I would not have referred to it at all but to show the depth of the pit to which we may fall if we do not know how to discriminate between fact and fiction. There are many kinds of evidence on which the antiquity of the human race is based and one principal type of it is the handiwork left by the primitive man. Extreme caution is necessary before the artifacts are accepted as such and to the want of this care is to be traced the mistaken recognition of pieces of stone fractured by natural agency as being due to artificial means. A remarkable case of the deceptive nature of a piece of rock was observed by me while in the Bhavanagar State[2] in Kathiawar in 1914 (Pl. I), where I came across a natural fragment of limestone simulating the peculiar shouldered types of neolithic celt first described from Burma [3] and afterwards from parts of Chutia-Nagpur [4] and of Assam.[5] In the Portuguese province of Sattary near Goa there is a petrified forest overlaid by the Deccan trap and laterite. According to Marchesetti [6] many of the trunks of the trees of which it is composed bear evident traces of the instruments which have been employed to cut them down. This will lead to the existence of a man in the pre-upper-Cretaceous time, but, as it has been shown by Dr. Ball,[7] the so-called evidence of cutting by the man is really due to the effect of jointing. Reference should be made in this connection to the interesting discussion that took place at the sitting of the Geological Society of

[1] A. C. Das., Rig-Vedic India, Vol. I, p. 21, 1921.

[2] Proc. Bang. Sahitya Parishad, Vol. 21, p. 36, 1321 B. S.

[3] Mem. Geol. Surv Ind., Vol. X, Pt. 2, pp. 167-171, 1874.

[4] Proc. Asiat. Soc. Beng., pp. 118-120, 1875.

[5] Journ. Asiat. Soc Beng., N. S., Vol. IX, pp. 291-293, 1913.

[6] Journ. Bomb. Br. R. A. S., Vol. XII, pp. 215-218, 1877.

[7] Mem. Geol. Sur. Ind., Vol. XIII, p. 218, 1877.

London held on the 19th November 1913[1] where Prof. Boyd Dawkins and others showed that often flints could be so broken by natural causes as to give rise to forms resembling exactly those produced artificially. It must also be borne in mind that unless the artificial implements are found in some natural deposits, they lose their intrinsic value as the zone-fossils of the Human period and that the correlation of the implements found at different places simply on the evidence of the chipping they have undergone is always open to a serious objection. To explain more clearly what I mean I wish to make a small digression.

Classification of Pre-historic Time.

The principle underlying the classification of the Pre-historic time was enunciated by Lucretius (99 or 98-55 B. C.) about two thousand years ago in a few significant lines which have been translated as below :—

"Arms of old were hand, nails and teeth and stones and boughs broken off from the forests, and flame and fire, as soon as they had become known. Afterwards the force of iron and copper was discovered; and the use of copper was known before that of iron." [2]

At present the anthropologists recognise a number of cultural stages into which the whole lithic period is divided and it may appear to a casual observer that the shape and the type of the implements are the factors which determine the stage to which any implement may be assigned. Nothing, however, is in reality far from truth and for a proper elucidation of this point I wish to give a very short account of the important pre-historic discoveries that were carried out at Menchecourt and Saint-Acheul in the neighbourhood of Abbeville and Amiens, respectively. Extremely interesting and far-reaching in their importance were the Pre-historic finds discovered at these places by M. Boucher de Perthes, one of the pioneer workers in the field of anthropology, in 1841, though the possibility of such a discovery occurred to him fifteen years before, *i. e.*, in 1826.[3] Important sections were discovered at these places where Palæolithic flint implements were found associated with mammalian and other fossils. For a complete account of these sections reference may be made to the contributions

[1] Q. J. G. S., Vol. 70, pp. ii-xii, 1914.

[2] Haddon : History of Anthropology. p 101, 1910.

[3] Phil. Trans. R. S., Vol. 150, p. 279, 1860.

to the Royal Society by Prestwich [1] and a paper published by Commont in 1908.[2] Two sections, as observed by Prestwich at Menchecourt, give very clearly an idea of the associates of the flint-implements. In his paper Commont has published some sections found at Saint-Acheul with a description which shows the arrangement of the beds with their peculiarly worked flint implements while an idea of the great antiquity of these beds may be made from an examination of the mammalian remains found at these places. In this connection attention may also be drawn to the great grotto of Castillo in northern Spain and to the stratigraphic section of the grotto with an appended description published by Osborn.[3] These show very clearly that the classification of the Palæolithic period into a number of cultural stages is chiefly based on the relative position of the implement-bearing beds together with the animal fossils while the shape and the nature of the chipping exhibited by the implements is of secondary importance. Mr. Johnson has published a figure containing four implements of the dumb-bell pattern.[4] A glance at the figure is quite sufficient to show the great clastic resemblance that exists between these specimens, but in point of age they range from the so-called Eolithic to the Neolithic. It is not the purpose of the present paper to enter into the question of the classification of the implements, but I wish to take an advantage of this opportunity to refer to a classificatory scheme recently proposed by Mr. Abbott which shows very clearly that the form of the chipped implements is not always a sure criterion of the particular cultural stage which is represented by it.[5]

Lower Siwalik Ancestor of Man.

One of the most perplexing questions is the date of the first appearance of man and in Burma we have evidences which may lead us to believe in the existence of an upper Tertiary man. Before treating this subject somewhat in detail I wish to refer to two other matters of great interest to the students of anthropology and one of them relates to the supposed Miocene relative of the

[1] Phil. Trans. R. S., Vol. 150, pp. 277-317, 1860.
[2] L'Anthrop., Tom. XIX, pp. 527-572, 1908.
[3] Osborn: Men of the Old Stone Age, p. 164, 1919.
[4] Folk Memory, p. 42, fig. 2, 1908.
[5] Sci. Progr., Vol. XII, pp. 272-281, 1917-1918.

Das-Gupta

PLATE I

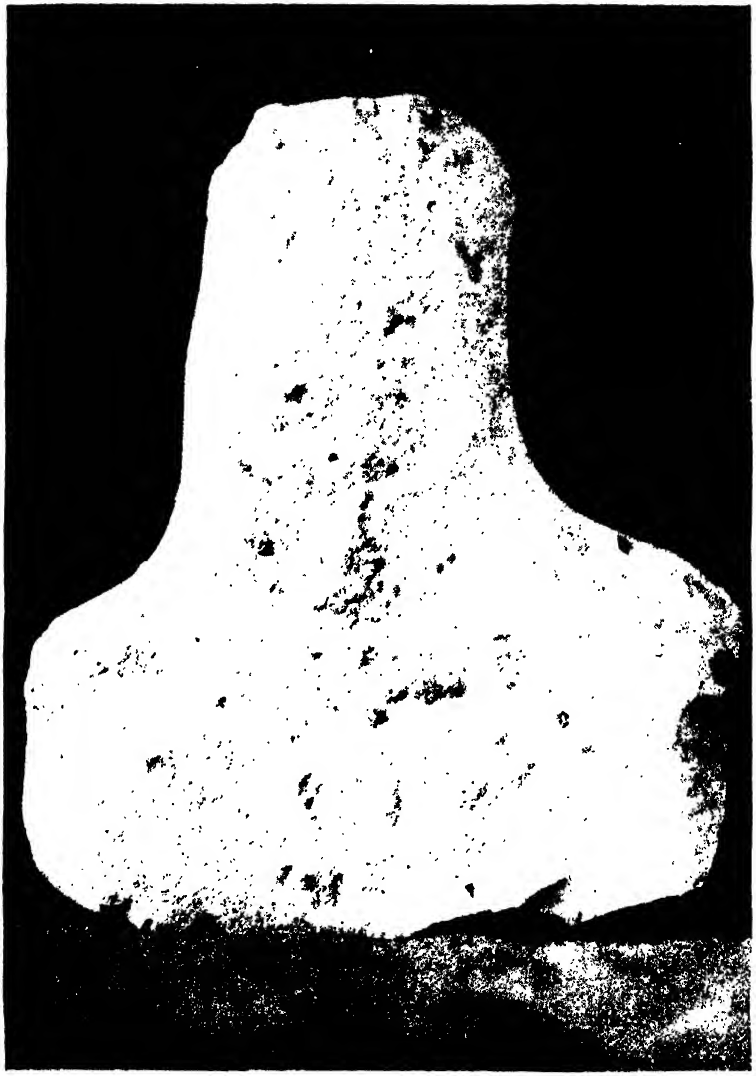


Photo by B. Maitra

PSEUDO-NEOLITH FROM BHAVANAGAR STATE,
KATHIAWAR

man. Recently a number of fossil Primates has been described from the Indian Siwalik beds[1] and they include the remains of an animal called *Sivapithecus indicus* by Dr. Pilgrim of the Geological Survey of India. This animal, found at Chinji, is known chiefly by the right mandibular ramus with the three molars (two of which are in a complete state), two pre-molars (one being in a good state of preservation) and a part of the canine. Besides this, a few other fragments have been found including the left half of another mandibular symphysis with the canine and the roots of the two incisors and of the first premolar. The most important human feature in them is the bicuspid nature of the last lower premolar, and, though there are many important points of discrepancy between the human mandible and that of a *Sivapithecus*, Dr. Pilgrim thinks that we may consider it possible that the human ancestor might have belonged to the same genus as the mandible of *Sivapithecus indicus*. According to the author, however, *Sivapithecus* is not on the main line of descent but a marginal adaptation. This paper has deservedly attracted a good deal of attention and according to M. Boule the evidence on which the above-mentioned conclusion is based is extremely weak.[2] According to Lydekker the animal is allied to *Palæopithecus* and not improbably they are generically identical.[3] Giuffrida-Ruggeri is also doubtful about the conclusion of Dr. Pilgrim.[4] Mr. W. K. Gregory[5] is of opinion that *Sivapithecus* is closely allied to *Dryopithecus* and is better referable to the *Simiidæ* than to the *Hominidæ*. His restoration of the jaw is also different from what was proposed by Dr. Pilgrim. The important points of distinction between these two reconstructions are given below :—

Pilgrim (1916).	Gregory (1916).
1. The mandible is wide.	1. The mandible is narrow.
2. The m_2 and m_3 have their internal edges touching the internal edges of the mandible.	2. The m_2 and m_3 have their internal edges at a distance of about 1·5 mm. from the internal edge of the mandible.

[1] Rec. Geol. Surv. Ind., Vol. XLV, pp. 1-74, 1916.

[2] L'Anthrop., Tom., XXVI, p. 409, 1915.

[3] Nature, Vol. 95, p. 277, 1915.

[4] Su l'origine dell'uomo, pp. 64-65, 1921.

[5] Bull. Amer. Mus. Nat. Hist., Vol. 35, pp. 287-293, 1916.

Pilgrim (1916).	Gregory (1916).
3. The distance between the internal edge of m_1 and the mandibular internal edge is 3 mm.	3. The distance between the internal edge of m_1 and the mandibular internal edge is about 1·5 mm.
4. The crown of pm_2 is situated within the internal and the external edges of the mandible.	4. A portion of the crown of pm_2 projects beyond the external edge of the mandible.
5. The distance between the internal edge of pm_1 and the internal edge of the mandible is 3 mm. and only a very small portion of its crown projects beyond the external edge of the mandible.	5. The distance between the internal edge of pm_1 and the internal edge of the mandible is 6 mm. and a very considerable portion of its crown projects beyond the external edge of the mandible.
6. The height of the manibular symphysis is 16·5 mm.	6. The height of the mandibular symphysis is 18 mm.
7. The incisors are internal to the canine so that in a side view they are not visible.	7. The incisors are external to the canine so that they are visible in a side view.
8. The crowns of the incisors touch one another.	8. There is a diastema between the incisors.

The materials on which these restorations are based are (1) the fragmentary right ramus found at Chinji, (2) the fragmentary left ramus found at Haritayalnagar and (3) the left upper canine from Chinji. Let us consider the points of distinction enumerated above in the light of the evidence supplied by the fossils themselves and see which restoration agrees more with the fossils. Here the numbers of the discrepancies are taken *seriatim*:—

1. Nothing decisive obtainable.

2. Dr. Pilgrim's restoration agrees more with the fig. 8 of pl. I.

3. The same remark as under (2).

4. The same remark as under (2).

5. Nothing decisive but more in favour of Dr. Pilgrim's restoration.

6. Dr. Pilgrim's restoration agrees more with fig. I of pl. 2, though the discrepancy is very small.

7. Mr. Gregory's restoration agrees more with the specimen as found in pl. 2, fig. I*a*.

8. The great diastema between the preserved portions of the incisors may lead one to believe that possibly Mr. Gregory's restoration is a more correct interpretation.

It thus appears that the evidences, though almost equibalanced, are more in favour of Dr. Pilgrim's restoration.

Dr. Pilgrim made an exhaustive comparison of *Sivapithecus indicus* with *Dryopithecus* and discussed the several points in which they differ from one another.[1] The pentacuspid arrangement of the molars and the bicuspid premolar give the fragments certainly a human aspect and it is most unfortunate that no description of the lower premolar of *Pithecanthropus erectus* has been published as it would certainly have thrown a good deal of light on this question. The lower premolar has got the human aspect in its bicuspid nature, but it differs from the modern man in being double-rooted. As remarked by M. Boule, the fossils are too fragmentary for any decisive conclusion, but it appears that, so far as the available materials are concerned, Dr. Pilgrim's deductions are more correct than those of Mr. Gregory. With the kindness of Dr. Pilgrim I had an access to the original specimens and it seems to me that in the right ramus there is one character which gives it a human aspect. The lower premolars are all two-rooted and the second molar shows that of its two roots, the anterior one is vertical while the posterior one is sloping backwards. This is indicative of a human type if we consider the following statement in connection with the human tooth :—[2]

'Each lower molar has two roots, an anterior, nearly vertical, and a posterior, directed obliquely backwards.'

The two-rooted premolar is very strange for a fossil which is supposed to be more or less in the ancestral line of man but the following remarks of Owen[3] are interesting in this connection :—

'Both upper and lower premolars are bicuspid......These teeth in both jaws are apparently implanted each by a single, long, subcompressed, conical fang; but that of the upper premolars is shown by the bifurcated pulp-cavity to be essentially two fangs, connate, and which, in some instances, are separate at their extremities.' It thus appears that *Sivapithecus indicus* combines in its mandible the human and the simian aspect in a very remarkable way and we may preferably

[1] *Op. cit.*, p. 49.

[2] Gray: Anatomy, p. 1041, 1916.

[3] Odontography, p. 453.

look upon it, at present, as belonging to the *Homosimiidæ*, the name being derived from *Homosimius*,[1] the supposed semi-human ancestor of the eoliths according to de Mortlett.

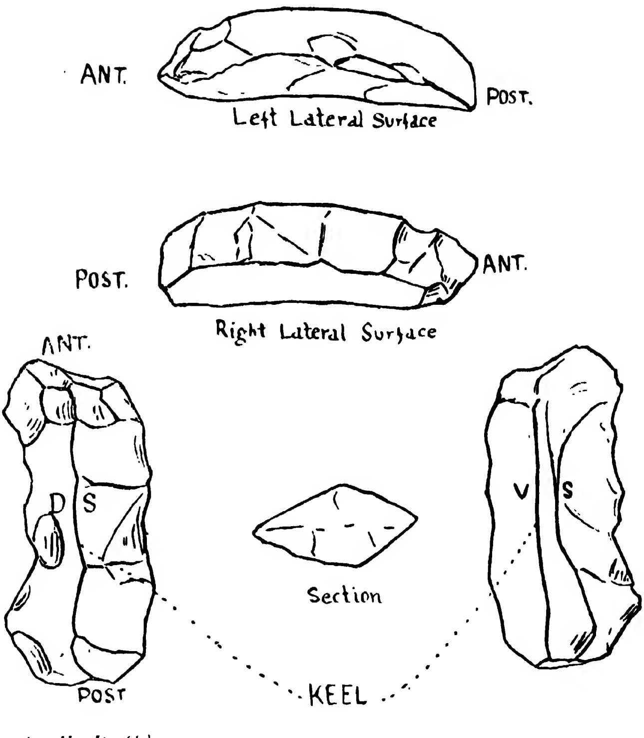
Tertiary Man.

The existence of the man in the Pleistocene time has been definitely established, but, as mentioned already, there is a good deal of controversy regarding his existence in the Tertiary period. According to M. Boule,[2] the existence of man in the lower Tertiary time appears to be nearly impossible, the existence of a man or rather of a pre-man is very possible in the Miocene time while his existence in the Pliocene time is quite probable. So far as the prevailing opinions are concerned, there are persons who seem to believe in the existence of a man in the Oligocene time and this theory is based not on any skeletal remains, but on the objects known as eoliths. There is a considerable difference of opinion regarding the origin of these objects. A school of anthropologists, including M. Rutot, regards them to be human artifacts and has proposed a number of cultural stages, *e.g.*—*Reutelian*, *Mafflian*, and *Mesvinian*—into which the Eolithic period may be divided. Sollas, Boule and others think that they are all due to natural causes while according to deMortillet they were made not by the man himself, but by a semi-human precursor which he named *Homosimius bourgeoissii*. M. Rutot thinks that these eoliths are comparable to the implements of the Tasmanians though there is a great difference in the materials of which they are composed.[3] Mention must be made in this connection to the very interesting observations made by M. Boule[4] at the cement works of Guerville, near Mantes and on the left bank of the Seine where, as remarked by M. Boule, 'every form of eolith is said to be produced in great numbers daily, as a by-product.'[5] According to the upholders of the natural origin of the eoliths, they are due to pressure, action of frost or that of a torrential river or any other natural cause. Another interesting observation to be mentioned in this connection is that due to the Abbe Breuil who has found unmistakable evidence of the breaking of a

[1] Sollas: Ancient Hunters, p. 55, 1911.
[2] M. Boule: Les Hommes fossiles, pp. 111-112, 1921.
[3] Bull. Soc. Belge. Geol., XXI, pp. 439-482, 1907.
[4] L'Anthrop., XVI, pp. 257-267, 1905.
[5] Sollas: Ancient Hunters, p. 66, 1911.

Das-Gupta

PLATE II



Drawn by K. D. C.]

DIFFERENT VIEWS OF A PALAEOLITH FROM BURMA
(ROSTRO-CARINATE TYPE, NATURAL SIZE)

piece of flint *in situ*, the broken fragments having all the appearances of the eoliths. In connection with this eolithic controversy attention may be drawn to another important point. All known eoliths are chiefly of flint, but the materials of the palæoliths and the neoliths may be of any rock. Quartz and its varieties are marked by a very characteristic conchoidal fracture and a great hardness. Obsidian has also been found to enter into the composition of the eoliths and this lava is likewise characterised by a prominent conchoidal frac'ure with a great hardness. I think that this explains why the eoliths are made up of flints or some other types of natural silica and obsidian because other rocks, when naturally fractured, will not show the previously mentioned conchoidal surfaces and will be less resisting to the weathering agencies. The Abbe Bourgeois found peculiarly crackled flint fragments in the upper Oligocene beds of Thenay and was of opinion that the crackling was the work of fire pointing to the existence of a man who used fire, but the crackling might have been due to a forest-fire as suggested by Dr. Noetling.[1]

It thus appears that the theory of the eoliths stands on very insecure grounds but, within recent years, another type of pre-Palæolithic objects has been found for which a human authorship has been claimed by Sir Ray Lankester and others. These have been called rostro-carinate implements by Lankester and, for a detailed study of them, reference should be made chiefly to the contributions of Sir Ray Lankester and J. R. Moir.[2] These implements are laterally compressed and represent the type of a pre-Palæolithic industry known as the Icenian industry. The beds in which they occur have been supposed by Sir Ray Lankester to be Pleistocene rather than Pliocene. Lankester has also recorded implements of the rostro-carinate type from the upper Miocene beds. The artificial nature of these implements has been questioned by Boule[3] and Prof. Sollas[4] and, according to the latter, "the rostro-carinate form is very simple and may be produced by chance blows. It arises whenever an elongated mass of flint—a nodule or a fragment already blocked out by joints—is traversed by two surfaces of fracture which are inclined in opposite directions and

[1] Centralbl. f. Min. Geol. u. Pal. pp. 748-753, 1908.

[2] Phil. Trans. R. S., Vol. 202 B, pp. 283-336, 1912, and Pre-palæolithic Man.

[3] L'Anthropologie, Vol. XXVI, p. 25, 1915.

[4] Rep. Brit. Assoc. Adv. Sci., pp. 788-790, 1914.

converge so as to intersect along a line (carina)." It appears further that the way in which Mr. Moir has tried to explain the variations in the rostro-carinates is only possible with a fairly well-developed type of intelligence which is not probable for a Heidelberg man as 'in most of the characters by which this species differs from modern men, it approaches the higher apes,'[1] though it should also be noted that, according to Dr. Keith, the mass of the brain of the Piltdown man attained the modern human standard so that 'the Piltdown man saw, heard, felt, thought and dreamt much as we still do.'[2] It may be mentioned here that Mr. Moir has recorded the occurrence of supposed rostro-carinate implements from a number of places, but in all cases they are made of flint and it appears to me that this similarity of the material goes a good deal against the theory that attributes them to an artificial origin.

Tertiary Man in Burma.

Though no eolith has been found in India, a few implements have been obtained in Burma from which the existence of a Tertiary man has been claimed by some and, as by a careful study of the available literature on the subject, I have been led to form an opinion contrary to that which is generally held, I shall be excused if more than a passing reference is made to the question because it is a matter of supreme importance from the anthropological point of view. In the year 1894 Dr. Noetling[3] published a paper in which he described the occurrence of chipped (?) flints in the Upper Miocene beds of Burma. In 1894 Prof. Rupert Jones[4] wrote a short review of the find and apparently supported Dr. Noetling. In 1895 Mr. Oldham[5] published a criticism of Dr. Noetling's paper and doubted the Tertiary age of the flakes. In 1897 Dr. Noetling[6] published a paper criticising Oldham in which he expressed an opinion that the age of the beds was to be considered as Pliocene while, in the same year, in another paper he described an artificially worn femur of a

[1] Sollas: Ancient Hunters, p. 50, 1911.
[2] Keith: The Antiquity of Man, p. 429, 1916.
[3] Rec. Geol. Surv. Ind., Vol 27, pp. 101-103, 1894.
[4] Nat. Sci., Vo . V, pp. 345-349, 1894.
[5] Nat. Sci., Vol. VII, pp. 201-202, 1895.
[6] Nat. Sci., Vol. X, pp. 233-241, 1897.

Hippopotamus from the lower Pliocene of Burma.[1] Two papers dealing with the subject were published by Swinhoe[2] who seems to differ from the view entertained by Dr. Noetling while, in his work dealing with the oil-fields of Burma,[3] Dr. Pascoe refers to the subject and appears to lean more on the side of those who do not wish to ascribe any value to Dr. Noetling's opinion. It is clear that to come to any conclusion regarding this matter we must discuss clearly the points at issue and they are the following :—

(1) Were the chips found *in situ* ?

(2) If the answer to the previous question is in the affirmative, what is the age of the beds in which the chips were found ?

(3) Are the chips of natural or of artificial origin ?

Regarding the point 1, there are two distinct opinions prevalent. According to Dr. Noetling the chips were found *in situ,* while Oldham thinks that they might have dropped from the overlying plateau. It is just and proper that every case of a reported discovery of a Tertiary man should be scrutinised with as much care as possible, but all examinations should be made with an open mind and not with that of a sceptic. According to the rejoinder published by Dr. Noetling to the criticism of Oldham, the latter gentleman is said to have spent only 15 minutes at the place of discovery while an examination of the section published by Noetling leads one to doubt the possibility of the falling down of the chips from the overlying plateau and lying embedded at the locality where they were found. The silence of Mr. Oldham and of the other explorers regarding this point apparently shows that the section published by Noetling represents the actual state of things. According to Oldham " the implements are not confined to the outcrop of the fossiliferous ferruginous bed, but are scattered over the surface of the plateau above " (*op. cit.*, p. 202). This fact and the apparent improbability of the existence of the Tertiary man led Mr. Oldham to doubt if the flakes were found *in situ.* The pieces of chert were found with mammalian remains ; never has any suggestion been made that they dropped down from the overlying plateau. Implements are found on the plateau above, but the " rectangular flake," to which more attention will be drawn later on, has not apparently got any representative in the overlying plateau ;

[1] Rec. Geol. Surv. Ind., Vol. 30, pp. 242-249, 1897.

[2] Zoologist Ser. 4, VI, pp. 321-336, 1902 and Ser. 4. VII, pp. 254-259, 1903.

[3] Mem. Geol. Surv. Ind., Vol. XL, Pt. I, pp. 53-54, 1912.

at least no one has described such a rectangular flake from this locality. A worn femur of *Hippopotamus iravaticus* was also detected embedded in a similar ferruginous layer only about a half mile from the place where the implements were found and, as will be shown later, there are strong reasons for attributing an artificial origin to these worn faces. I have not seen the place myself, so it is not possible for me to make any positive assertion on the matter, but it appears to me from an examination of the literature on the subject that there are very strong reasons for thinking that the chert pieces were found *in situ*. It may be mentioned that this opinion is based not on the strength of any one particular evidence, but on the strength of so many evidences which all point to the existence of such an early handiwork of a man.

Let us now turn our attention to the second point. According to the opinion originally expressed by Dr. Noetling the age of the ferruginous conglomerate in which the chips were found was either Pliocene or Upper Miocene (*op. cit.*, p. 102). Afterwards he pronounced more definitely upon the age of this conglomerate and put it down as Pliocene.[1] The beds in which the specimens were found overlie the Pegu series of Burma and have been described as the Irrawadian series by Dr. Pascoe, the series consisting of three distinct members. The whole stratigraphic scheme, according to Dr. Pascoe (*op. cit.*, Pl. 8), is as given below :—

Recent stream alluvium.

Plateau deposits

Irrawadian series :—	Irrawadian sandstones, etc. Ferruginous conglomerates. Red bed.

Pegu series.

It is in the ferruginous conglomerates that the specimens have been observed and "it is in these conglomerates that the vertebrate remains of Siwalik affinities have been found."[2] Dr. Pascoe has correlated the Irrawadian series with the Pliocene including the Pontian. According to Dr. Pilgrim the mammalian remains of the Irrawadian series point towards the middle Siwalik age which corresponds with the Pontian stage of Europe.[3] The following mammalian remains have been recorded from this conglomerate, the list being compiled from the names given by Noetling, Pilgrim and Pascoe :—

[1] Nat. Sci., Vol. X, p. 234, 1897.

[2] Mem. Geol. Surv. Ind., Vol. XL, Pt. I, p. 33, 1912.

[3] Rec. Geol. Surv. Ind., Vol. XLIII, Pl. 26, 1913.

+(1) *Mastodon Sivalensis.*
+(2) *M. latidens.*
+(3) *Stegodon clifti.*
+(4) *Aceratherium Lydekkeri.*
+(5) *A. perimense.*
+(6) *Rhinoceros sivalensis.*
+(7) *Hipparion panjabiense.*
+(8) *Hippodactylus antilopinum.*
+(9) *Merycopotamus dissimilis*
(10) *Tetraconodon minor.*
+(11) *Sus titan.*
+(12) *Hippopotamus sivalensis.*
+(13) *H. iravaticus.*
(14) *Dorcatherium birmanicum.*
+(15) *Vishnutherium iravaticum.*
(16) *Hydaspitherium biramanicum.*
+(17) *Taurotragus latidens.*

In the foregoing list the species marked with a + have also been found outside Burma. Their geological distribution, according to Dr. Pilgrim, is given below :—

Fossils.	L.S.	M.S.	U.S.
1. *Mastodon sivalensis*	...	...	+
2. *M. Latidens*	...	+	...
3. *Stegodon clifti*	...	+	...
4. *Aceratherium Lydekkeri*	...	+	...
5. *A. perimense*	+	...	...
6. *Rhinoceros sivalensis*	...	+	+
7. *Hipparion panjabiense*	...	+	...
8. *Hippodactylus antilopinum*	...	...	+
9. *Merycopotamus dissimilis*	...	+	+
10. *Sus titan*	...	+	...
11. *Hippopotamus sivalensis*	...	...	+
12. *H. Irvaticus*	...	+	...
13. *Vishnutherium iravaticum*	...	+	...
14. *Taurotragus latidens*	...	+	...

Thus, it is clear, as has been pointed out by Dr. Pilgrim, that the Burma mammalian fossils occurring in the neighbourhood of Yenangyoung show a characteristic middle Siwalik facies which is supposed to be the representative of the Pontian stage of Europe.

Now remains the third point and in connection with this I wish at first to refer to the worn femur of *Hippopotamus iravaticus* in which "facets are exhibited on the anterior and posterior side of both extremities in such a way that they run parallel to the axis of the shaft; no faces or any other traces of wear and tear are either noticed on the shaft or on the proximal or distal face." In his paper Dr. Noetling discussed all possible causes which might have produced the facets, but found that the best way in which the manufacture of the facets might be explained was by ascribing them to some artificial *i.e.* anthropoid agency. I think that this conclusion has since received a strong support from an observation of Mr. Swinhoe to whom was brought the upper premolar of a small fossil Rhinoceros with facets on three separate prominences which, on the admission of the man who brought it, were produced by the man himself. Accordingly, for all we know hitherto, there is nothing to preclude the probability that the facets on the femur of *Hippopotamus iravaticus* were produced in the way explained by Dr. Noetling. When the chips are taken into consideration there appears to be a strong reason for ascribing them to an artificial origin. With the permission of the authorities of the Geological Survey of India, I had an opportunity of examining these chips and it appears to me that, but for the presence of the "rectangular flake," it would not have been possible to come to a definite conclusion regarding their origin. This chip, which has also been figured by Prof. Rupert Jones, is extremely striking as it has got faces which must have been produced artificially. The accompanying plate (Pl. II) contains the drawings of this implement and for these drawings I am extremely thankful to Mr. Tipper of the Geological Survey of India. Dr. Noetling has compared this implement with the one found in the Godavery valley, but it appears to me that it may be better looked upon as representing a rostro-carinate type. The simplest form of a rostro-carinate implement exhibits (1) an anterior, (2) a posterior, (3) a dorsal, (4) a ventral, and (5) two lateral surfaces with a keel on the dorsal aspect. The implement from Yenangyoung is a little more complex than the simplest pattern and in it there can be very easily

distinguished a ventral and a dorsal aspect with a keel on each side, the keel on the ventral aspect being not so perfect as that on the dorsal side. Following Mr. Moir the side which has a less declivity is recognised as the anterior side, so that the right lateral surface shows faces of which three are markedly rectangular. The left lateral surface shows two small faces one lying over another at the anterior end. On the ventral side there is a small triangular area in the middle which may be described as the ventral surface. The section is of the rhomboidal type. The specimen may be compared with the implement figured by Moir in plate 9 of his Prepalæolithic man, the profile of which exhibits "a marked resemblance to the profile of the rostro-carinate implement" (*op. cit.*, p. 37). Thus, it is clear that we are here dealing not only with a human artifact, but with an implement which is possibly more primitive in pattern than the forms which are usually regarded as palæoliths and that there is not only nothing to doubt the artificial nature of the implements, but the nature of at least one of them shows unmistakably that it represents a cultural stage which, if not pre-Palæolithic, is representative of the earliest Palæolithic type and we have in Burma evidences which may probably point to the existence of a man in the middle Siwalik or Pontian time.

Alluvial Man.

Hitherto I have been dealing with the cases of human handicraft about the geological position of which some doubt is maintained in some quarters, but in the present section I wish to discuss the cases which enable us to fix the age of the artifacts with more precision because they were found associated with mammalian and other remains and their mode of occurrence was so clear as to preclude the possibility of any doubt as to the nature of the beds from which they were obtained. By the fortunate discovery of implements found associated with mammalian remains it has been possible to work out the different cultural stages of the Palæolithic period in the west, but unfortunately the number of such finds in India is very limited and it is my purpose to describe these finds, *i.e.*, one found by Hacket at Bhutra in the Narsinghpur district on the left bank of the Narbada and the other by Wynne below the village of Mungi (Hyderabad) on the bank of the Godaveri. The Mungi specimen was found

earlier and, besides being mentioned at a meeting of the Asiatic Society of Bengal,[1] we also find it referred to by Blanford[2] and by Wynne,[3] while a detailed description of it was given by Dr. Oldham.[4] Hacket's find was described by Medlicott[5] while Theobold dealt with the shells found in the ossiferous gravels of the Narbada.[6] Dr. Oldham put the Godavery gravels under the Pliocene while Medlicott pointed out that the Narbada ossiferous deposits were not older than the late Pleistocene. It is admitted by all that the implement-bearing old alluvia of the Godavery and of the Narbada are of the same age and the determination of the age of these deposits being so important from an anthropological point of view, let us examine the materials on the evidence of which their age may be fixed.

Both these deposits contain mammalian remains and the Narbada deposits are richer than the Godavery ones and the following fossils have been listed from the Narbada alluvium [7] :—

(1) *Ursus namadicus.*
(2) *Bos namadicus.*
(3) *Bos palæogaurus.*
+(4) *Bubalus palæindicus.*

[1] Proc. Asia Soc. Beng., pp. 207-208, 1865.
[2] Geol. Mag., Vol. III, pp. 93-94, 1866.
[3] Geol. Mag., Vol. III, pp. 283-284, 1866.
[4] Rec. Geol. Surv. Ind., Vol. I, pp. 65-69, 1868.
[5] Rec. Geol. Surv. Ind., Vol. VI, pp. 49-54, 1873.
[6] *Ibid*, pp. 54-47.
[7] Pal. Ind., Ser. X, Vol. III, pp. 125 *et seq.* The existence of *Bos palæogaurus* in these beds has been mentioned by Flower and Lydekker (Mammals Living and Extinct, p. 366). *Bubalus palæindicus* is possibly a variety of B. *buffelus* (N. J. F. Min. Geol. u. Pal. Bd. 1, p. 119, 1909).

In two papers contributed to the Journal of the Department of Letters, Calcutta University (Vol. I, pp. 113-200 and Vol. III, pp. 159-224, 1920), Prof. Panchanan Mitra has given a summary of pre-historic Indian culture. In one of this (Vol. III, p. 167) it has been stated that *Giraffa* has been found in the older alluvium of India. I do not know Prof. Mitra's authority for such a statement. There are few other points in these two papers that I am unable to follow. Thus, for example, in his paper published in Vol. I, he has made an attempt to classify the Indian palæoliths into a number of cultural stages, but this attempt has been preceded by the statement that "one day it might be found more suitable to name the stages in India as "Post-Siwalik" "Pre-Narbada" "Early-Sabarmati," "Late-Sabarmati," etc." (p. 124). It appears to me that here Prof. Mitra has reversed the usual procedure.

(5) *Leptobos Frazeri.*
(6) *Boselephas namadicus.*
+ +(7) *Cervus*, sp. (? *C. duvaucelli*).
(8) *Sus*, sp.
(9) *Hippopotamus palæindicus.*
(10) *Hippopotamus namadicus.*
+(11) *Equus namadicus.*
+ +(12) *Rhinoceros unicornis.*
(13) *Elephas namadicus.*
+(14) *Elephas insignis.*
+(15) *Elephas Ganesa.*

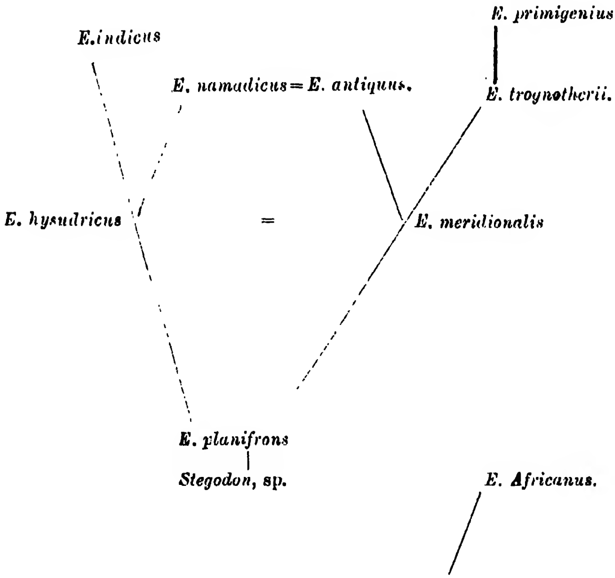
The species marked with + are those that have been found only in the Siwaliks while those marked with + + are found living now. As remarked before, the Godavary beds are very poor in fossils and they do not contain any species unknown in the Narbada beds. In 1904 fossil bones were found by Mr. Beale at Nandur Madmeshwar on the Godavary river and the bed, on examination by Dr. Pilgrim, yielded an elephant skull and, from a study of this skull, Dr. Pilgrim came to the conclusion that the Narbada elephant was only a variety of *Elephas antiquus.*[1] It may be added that the existence of *Elephas namadicus* from the Godavary alluvium had been noted previously also. Dr. Pilgrim also obtained from the Godavary beds a single tooth of *Equus namadicus* and the lower jaw, possibly, of *Hippopotamus palæindicus.* Thus it is quite clear that the Godavary and the Narbada ossiferous alluvia are contemporaneous and we shall now proceed to find out their age.

In 1900 the late Dr. Blanford published a paper dealing with the distribution of the vertebrate fauna found in India[2] and it appears that some genera found as fossils in the alluvial beds of the Godavary and the Narbada are not known to occur, at present, in the regions including these alluvia and this points to a great antiquity of the deposits. If, however, the fossils are compared with the upper Siwalik forms, it is found that, though the upper Siwalik beds contain a few genera that may be living now, no upper Siwalik species has continued up to the recent time. There is also a great structural unconformity between these alluvial beds and the upper

[1] Rec. Geol. Surv. Ind., Vol. 32, pp. 199-211, 1905.
[2] Phil. Trans. Roy. Soc., Vol. 194-B. pp. 335-436, 1900.

Siwaliks. Accordingly the age of these alluvial beds may be fixed as Pleistocene. The Pleistocene beds are divided into upper, middle and lower and we shall now proceed to see to what part of the Pleistocene time these alluvial beds may be relegated.

According to Medlicott these deposits are possibly not older than the late Pleistocene, but the extremely small number of living species in these deposits makes us hesitate to accept this view. Dr. Pilgrim's conclusion about the specific identity between *E. namadicus* and *E. antiquus* has been referred to already. This identity had been previously suggested by Leith Adams and others, while it has also been suggested by Soergel[1] according to whom the genealogy of the Narbada elephant may be expressed in the manner indicated below :—



It thus appears that Dr. Pilgrim's opinion that the Narbada elephant was migratory from Europe is contradicted. An attempt to fix the age of the Narbada deposits with the help of the distribution of *E. antiquus* in Europe does not appear to be very promising. According to M. Boule, this species is characteristic of the inferior Pleistocene

[1] Palæontographica, Vol. 60, p. 99, 1913. It has been mentioned by Dr. Scharff The history of the European Fauna, p. 252, 1899) that there is a great and marked similarity between *E. meridionalis and E. hysudricus*.

and is replaced by the mammoth in the middle Pleistocene,[1] but very likely this statement requires to be corrected as, according to Soergel,[2] the species was found throughout the whole of the Pleistocene time, and in the fourth glacial stage (Würmeiszeit) it lived in Italy and in France by the side of *E. primigenius*. Dr. Pilgrim's opinion that these deposits are not earlier than the lower Pleistocene is based on the supposed migratory character of the Narbada elephant but, as we have seen already, this opinion has been questioned as it is supposed to be derived from *E. hysudricus*, a well-known Siwalik species.

I have examined carefully the case of the other fossil species and it appears to me that if we analyse the evidence of the fossil species of *Hippopotamus* we may come to a conclusion somewhat more definite. The family of *Hippopotamidæ* is now confined to Africa where it is represented by two species, *H. amphibius* and *H. liberiensis*. The dental formula of *H. amphibius* is $i\,\frac{2}{2}$, $c\,\frac{1}{1}$, $pm\,\frac{4}{4}$, $m\,\frac{3}{3}=40$ and of *H. liberiensis*, is $i\,\frac{1}{1}$, $c\,\frac{1}{1}$, $pm\,\frac{4}{4}$, $m\,\frac{3}{3}=36$. Though the family is poorly represented and so limited in geographical distribution now, it was very widely represented in the Tertiary beds. The genus *Hippopotamus* begins in the Pliocene time and remains of the genus have been found in Asia and Europe besides Africa. Four species of the genus have been recorded from the Indian beds and of them one (*H. iravaticus*) comes from the middle Siwalik, one (*H. sivalensis*) from the upper Siwalik and two (*H. namadicus* and *H. palæindicus*) from the Pleistocene beds of the Narbada and the latter of the last two possibly from the Pleistocene beds of the Godavary.

According to Falconer two sub-genera can be distinguished in the genus *Hippopotamus*, namely *Hexaprotodon* and *Tetraprotodon*. The recognition of *Hexaprotodon* and *Tetraprotodon* as possessing sub-generic value has been questioned, but they undoubtedly form two distinct sections of this genus and the arrangement and the number of the incisors in this genus in the different species are so well-marked that from a study of the Indian species, the line of the specialisation of the genus can be very well observed and according to Lydekker[3] this specialisation has proceeded chiefly on two main lines :—(*i*) the shortening and widening of the mandibular symphysis,

[1] M. Boule : Les Hommes fossiles, p. 52, 1921.

[2] *Op. cit.*, p. 110.

[3] Pal. Ind., Ser. X, Vol. III, Pt. 2, pp. 47-48, 1884.

frequently accompanied by a general shortening of the cranium and the mandible and (*ii*) the reduction of the number of the incisors, the reduction being accompanied with the largely increased size of one or more of the remaining pairs of the incisors while the canines are also increased in size. Thus it is highly probable that the middle Siwalik *H. iravaticus*, the upper Siwalik *H. sivalensis* and the Narbada *H. namadicus* and *H. palæindicus* are in one actual line of descent. *H. iravaticus* has a long and narrow symphysis, 6 small incisors and a small canine; the next step is *H. sivalensis* with a considerably shorter symphysis but the same type of incisors; the third step is *H. namadicus* in which the symphysis is still more decreased in length, while $\bar{i}_2$ has become slightly smaller and is thrown more or less above the line of the other two. The fourth step is *H. palæindicus* in which the symphysis is shorter than in any other form, while $\bar{i}_1$ and $\bar{i}_3$ have increased considerably at the expense of $\bar{i}_2$ which becomes wedged in between the other two.

From the above description it becomes quite clear that the *Hexaprotodont* type is more archaic than the *Tetraprotodont* one and let us now proceed to find out the upper limit of the former group. As is known at present, besides the Indian occurrences, the *Hexaprotodont* type has been met with in the upper Tertiary beds of Algeria and in the *Pithecanthropus*-bearing beds of Java. This Java Hippopotamus—*H. sivajavanicus*, Dubois—is closely related to *H. sivalensis* and distinct from the Narbada species.[1] From this we can safely conclude that the Narbada alluvia are younger than the beds containing *Pithecanthropus* and let us now see what we know regarding the age of these beds of Trinil.

In course of his discussion of the age of the materials obtained by the Selenka-Trinil expedition Prof. M. Blanckenhorn has mentioned that there are three different groups of authors holding different views regarding the age of these beds.[2] According to one group the age is Pliocene, according to a second group it is diluvial or Pleistocene while a third group leaves the question still undecided (die Frage unentschieden lässt). According to Volz the beds may be middle diluvial in age.[3] It is known to some of us that according to Mr. La Touche the relics of a Pleistocene ice age in the plains

[1] N. J. f. Min. Geol. u. Pal. Bd. I, p. 119, 1909.
[2] Die Pithecanthropus-Schichten auf Java, p. 205, 1911.
[3] N. J. f. Min Geol. u. Pal. Festband., p. 270, 1907.

of Northern India are exhibited by the older alluvia which rise to a considerable height above the present flood-levels.[1] Prof. Blanckenhorn thinks that the time of the formation of the *Pithecanthropus*-bearing beds was characterised by an extremely heavy downpour,[2] an opinion shared by a few other geologists including Volz.[3] According to Prof. Blanckenhorn this pluvial time may be divided into three periods, the first corresponding to the Günz ice-age, the middle to the first short interglacial and the third to the Mindel ice period. As regards the age of the beds the author concludes that it may be either very late Pliocene or very early diluvial. Judging the question from the evidence of *Hippopotamus* fossils we also arrive at almost the same conclusions for the Javanese species which is related with the upper Siwalik *H. sivalensis* cannot be younger than the oldest Pleistocene, while the Narbada species of *Hippopotamus* is younger than the oldest Pleistocene as it really shows in its mandible a stage of transition between the *Hexaprotodont* and the *Tetraprotodont* type and I think that, under these circumstances, we may ascribe a middle Pleistocene age to the Narbada beds. It may be argued that the Narbada Hippopotamus and the Trinil Hippopotamus were contemporary and, as a matter of fact, the Trinil fauna consists of *Stegodon Ganesa var. Javanica* and *Elephas, sp. cf. antiquus*. We have seen, however, that the Narbada fauna consists of a few mammalian species which are living now, while the Java fauna does not include any such mammal.[4] It is thus quite clear that the Narbada beds are younger than the beds containing *Pithecanthropus*. The Narbada fossils include a number of freshwater molluscs and, according to Theobald, these are all represented in the living fauna, though, as it appears from the measurements given by Theobald, they are undergoing a little decadence. This fact may lead us to go up in the scale and to fix the age of the Narbada and the Godavary alluvia as being upper Pleistocene, but a little consideration will show that this is not possible.

Besides the Narbada and the Godavary, mammaliferous older alluvia have been found in the beds of the Kistna, the Jumna and the Ganges. The Kistna alluvium has yielded *Rhinoceros deccanaensis*,

[1] Geol. Mag., pp. 193-201, 1910.
[2] *Op. cit.*, p. 263.
[3] *Op. cit.*, p. 267.
[4] Pithecanthropus-Schichten auf Java, p. 143, 1911.

a few bovine teeth, a few shells, and probably a Mastodon described as *M. pandionis*. The Jumna and the Ganges older alluvia include the following mammalian fossils :—[1]

1. *Semnopithecus*, sp.
2. *Elephas antiquus*.
3. *Mus*, sp.
4. *Hippopotamus palæindicus*.
5. *Equus*, sp.
6. *Sus*, sp.
7. *Bos palæindicus*.
8. *Bos cf. namadicus*.
9. *Bos*, sp.
10. *Antilope*, sp.
11. *Cervus*, sp.

I wish to leave the Tapti alluvium alone, as it has nothing to do with the purpose of the present paper. From a consideration of the Jumna alluvial fossils, Mr. Oldham concluded that these beds were newer than those containing the Narbada remains. This conclusion has since been corroborated by Dr. Pilgrim who detected in the *Bos cf. namadicus* certain structural variations tending towards the recent species of Bos. Accordingly it appears more likely that the upper Pleistocene stage is represented by the older Jumno-Gangetic alluvium and as this alluvium is younger than the Narbada Pleistocene deposits, we may reasonably relegate the Narbada and the Godavary alluvial beds to the middle Pleistocene. This is further corroborated by an opinion recently expressed by Dr. Pilgrim that the topmost Siwalik is contemporaneous with the lowest Pleistocene.[2]

Thus the unmistakable evidence about the existence of man in India can be traced down to the middle Pleistocene.

It may be mentioned in this connection that palæoliths have also been found at a few other places in India in alluvial deposits. One of the latest of such finds is in the Kanhan valley[3] where two alluvia have been found with palæolithic and neolithic implements by Dr. Fermor. In these cases no associated animal remains have been

[1] Oldham : Geology of India. p. 436, 1893 and Rec. Geol. Surv. Ind., Vol 31, pp. 176-177, 1904.

[2] Journ. Asiat. Soc. Beng., N. S., Vol. XV, p. 85, 1919.

Rec, Geol, Surv. Ind., Vol. 47, p. 36, 1916.

found and, accordingly, at the present stage of our knowledge of Indian stratigraphy, it is not possible to determine the age of these implements. Thus, for example, many of the affluents of the Kistna contain palæolithic implements but, as remarked by Mr. Oldham, "the relations between the ossiferous gravels and those containing the implements are, however, somewhat obscure."[1] Nothing definite is also possible in the cases of the implements found in the detrital laterites because the age of these lateritic deposits has not been ascertained with any proximity to accuracy. These lateritic deposits have been developed extensively along the eastern coast of the Peninsula and they overlie the Cuddalore sandstone the age of which is extremely doubtful as no fossil has been found in it excepting the Trivicary wood. The general prevalent idea is that the low-level detrital laterites are post-Tertiary in age, while a section, found in the State of Mayurbhanj, of laterite with an under-lying bed of white clay which is again underlain by a bed of limestone[2] with *Ostrea*, foraminifera and fish teeth of an upper Tertiary facies also points to the post-Tertiary age of these lateritic deposits. But possibly all the detrital lateritic deposits were not formed at the same cultural period because though a number of palæolithic implements has been found associated with these laterites, pottery fragments which signify a neolithic culture have also been found embedded in them. About the relative age of the different lateritic deposits, Oldham thinks that "it is probable that the land rose very slowly from the sea, the laterite forming on the raised slope *pari passu* with the elevation, and that, consequently, the farther inland the rock the older its date." [3]

Cave Man.

We know that in Europe the palæolithic man lived in caves and the antiquity of the cave man has been determined by the bones of the extinct animals left in the caves and also by the relics of the work of art left by him. As pointed out by Prestwich, while dealing

[1] Geology of India, p. 403.

[2] Rec. Geol. Surv. Ind., Vol. 31, p. 167, 1904; Cal. Univ. Inst. Mag., pp. 63-67, 1906; Rec. Geol. Surv. Ind., Vol. 34, p. 135, 1906 and Rec. Geol. Surv. Ind., Vol. 41, p. 63, 1911.

[3] Geology of India, p. 389.

with the cave deposits, one should be careful that no terrestrial disturbance had helped in mingling together the deposits lying originally on two floors, one overlying the other.[1] In the countries marked by periods of intense glaciation during the Pleistocene time, the man very often took shelter in caves to save himself from the rigours of the arctic climate and, as evidence is lacking, in India, of such an intensely glaciated Pleistocene time it is no wonder that the deposits left by the primitive Indians inside the caves are really scanty. According to Logan[2] the cause of the smallness of our knowledge of the cave deposits in India is "not the want of caves in India, but of people to explore them," but it appears to me that, though it is true that the number of cave explorers in India is surprisingly small, we can never hope to have anything like the western record of the ossiferous caves with human remains. Only in two different parts of India have caves been found with animal remains in them, *viz.*, in the Presidency of Madras and in Kashmir. The Madras cave deposits will be dealt with first as they are by far the more important. These caves were at first explored by Captain Newbold[3] and appear to have been completely forgotten afterwards. Subsequently at the suggestion of Prof. Huxley, the Right Honourable M. E. Grant Duff, the then Governor of Madras, arranged for the exploration of the caves and the work was done by Messrs. R. B. Foote and H. B. Foote.[4] A number of fossils was found and they were described by Lydekker.[5] A number of caves was explored, but, from our present point of view, the caves at Billa Surgam are important. This place is within the Nandiyal taluq of the district of Karnul and in the neighbourhood of Betumcherru. There is a number of caves in the area, namely the Charnel House, the Purgatory, the North Chapel, the Hermit's Cell, the Cathedral, the South Chapel, and the Chapter House. Human teeth were found in the Charnel House Cave, while a number of bone implements has also been obtained.

Besides these a number of other fossils has also been met with and let us turn to them to fix the age of the deposits. The Karnul cave

[1] Phil. Trans. Roy. Soc., p. 280, 1860.

[2] Old Chipped Stones of India, p. 37, 1906.

[3] Journ. Asia. Soc. Beng., Vol. XIII, pp. 610-611, 1844.

[4] Rec. Geol. Surv. Ind., Vol. 17, pp. 27-31 and pp. 200-203, 1884; *ibid.*, Vol. 18, pp. 227-235, 1885.

[5] Rec. Geol. Surv. Ind., Vol. 19, pp. 120-122, 1886, and Pal. Ind., Ser. X, Vol. IV, Pt. 2, pp. 23-58, 1886.

fossils include molluscs, amphibia, reptiles, birds and mammals. The molluscs are represented by species which are living now and a similar remark is applicable to the other classes of animals excepting the mammals, the nature of which w ll accordingly help us in fixing the age of the deposits with a ertain amount of precision. The following list gives us an account of the fossil mammals found in the Karnul caves with some notes about them arranged in a tabulated form.

Fossil species.	Cave.	Living or extinct.	Remarks.
1. *Semnopithecus entellus*, Dufr: var. (The Hanuman monkey.)	Cathedral	Living	Found in Southern India.
2. *Cynocephalus*, sp. (A type of monkey.)	Charnel House.		The affinity of this is doubtful, but it is very closely allied to the African *C. anbis*. This genus is not found in India.
3. *Felis tigris* (*or ? leo*), Linn. (The tiger or lion.)	Purgatory and Cathedral.	Living	The tiger is found throughout the whole of India, while the distribution of the lion is limited, it being found at present in Kathiawar and probably also in the wildest parts of Rajputana.
4. (?) *F. pardus*, Linn. (The leopard.)	Cathedral	Living	Found practically throughout the whole of India.
5. *F. chaus*, Güld (The jungle cat)	Cathedral	Living	Very widely distributed throughout the whole of India.
6. *F. rubiginosa*, Geoffr. (The rusty-spotted cat.)	Cathedral	Living	Found in Southern India and Ceylon.
7. *Hyæna crocuta*, Erxl.	Cathedral	Living	This species has been found living in Africa.
8. *Viverra karnuliensis*, Lyd. (A type of civet.)	Charnel House.	Extinct	
9. *Prionodon* (?) sp. (A type of tiger-civet.)	Cathedral		Though the generic determination is doubtful, it is unlike any animal now living in Southern India.

Fossil species.	Cave.	Living or extinct.	REMARKS.
10. *Herpestes mungo*, Gmel. (The common Indian mungoose.)	Cathedral and Purgatory.	Living	Found throughout the whole of India. Possibly the specimens are of later age than the beds in which they occur.
11. *H. fuscus*, Waterhouse (The Nilgiri brown mungoose.)	Cathedral	Living	Found in Southern India. Possibly another peninsular species (*H. Smithi*, Gray) was also found in this cave.
12. *H. auropunctatus*, Hodgs. (The small Indian mungoose.)	Cathedral	Living	Found throughout Northern India and not found in the Peninsular portion.
13. *Melursus ursinus*, Shaw. (The Indian bear.)	Chapter-House.	Living	Found throughout the whole of India.
14 *Sorex*, sp. (A type of shrew.)	Cathedral and Charnel House.		This genus is widely distributed in India.
15. *Taphozous saccolæmus*, Temm. (The pouch-bearing sheath tailed bat.)	Cathedral and Charnel House.	Living	Found in many parts of India including the Madras Presidency.
16. *Hipposiderus diadema*, Geoffr. (The large Malaya leaf-nosed bat.)	Cathedral and Charnel House.	Living	Possibly occurs in the Madras Presidency.
17. *Sciurus bicolor*, Sparr (The large Malay squirrel.)	Cathedral	Living	Not found in Southern India.
18. *Gerbillus indicus*, Hard. (The Indian antelope rat.)	Cathedral and Charnel House.	Living	Found throughout India.
19. *Nesocia bandicota*, Bech. (The bandicoot-rat.)	Charnel House.	Living	Found in Southern India.
20. *N. bengalensis*, Gray and Hard. (The Indian mole-rat.)	Cathedral and Charnel House.	Living	Found practically throughout the whole of India.
21. *Mus mettada*, Gray. (The soft-furred field-rat.)	Cathedral and Charnel House.	Living	Found in several parts of the Madras Presidency.

Fossil species.	Cave.	Living or extinct.	REMARKS.
22. *M. platythrix*, Bennett. (The brown spiny mouse.)	Charnel House.	Living	Found in Malabar, Sind and the Punjab The fresh condition of the cave specimens points to their recent introduction.
23. *Golunda ellioti*, Gray. (The Indian bush-rat.)	Charnel House.	Living	Found in several parts of the Madras Presidency. The cave specimens are possibly of recent origin.
24. *Hystrix crasidens*, Lyd. (A type of porcupine.)	Cathedral	Extinct	......
25. *Atherura karnulieusis*, Lyd. (A type of porcupine.)	Cathedral	Extinct	This genus is not found in the Madras Presidency now.
26. *Lepus* cf. *nigricollis*, Cuv. (The black-naped hare.)	Cathedral, Charnel House and Purgatory.		......
27. *Equus asinus*, Linn. (A type of horse.)	Cathedral	Living	This species is now confined to North Africa. There is another type of horse possibly allied to some South African type.
28. *Rhinoceros karnulienis*, Lyd.	Cathedral and Charnel House.	Extinct	......
29. *Boselephas tragocamelus*, Pallas. (The Nilgai.)	Cathedral, Chapter House and Charnel House.	Living	Found practically throughout the whole of India.
30. *Gazella bennetti*, Sykes. (The Indian gazelle.)	Cathedral, Charnel House and Purgatory (?)	Living	Found in many parts of India including the Madras Presidency.
31. *Antilope cervicapra*, Linn. (The Indian antelope.)	Cathedral	Living	Found throughout the whole of India.
32. *Tetracerus quadricornis*, Blainv. (The four-horned antelope.)	Cathedral	Living	Found in parts of the Madras Presidency.

Fossil species.	Cave.	Living or extinct.	REMARKS.
33. *Cervus unicolor*, Bech. (The Sambar.)	Cathedral and Charnel Houss.	Living	Found throughout the whole of India.
34. *C. axis*, Erxl. (The spotted deer.)	Cathedral	Living	Found nearly throughout the whole of India.
35 (?) *Cervulus muntjac*, Zimm. (The barking deer.)	Purgatory	Living	Found throughout India.
36. *Tragulus cf. meminna*, Erxl. (The Indian Chevrotain.)	Cathedral		.....
37. *Sus cristatus*, Wagner. (The Indian wild boar.)	Cathedral	Living	Found throughout India.
38. *S. karnuliensis*, Lyd. (A type of boar.)	Cathedral	Extinct	.....
39. *Manis gigantea*, Illiger. (A type of pangolin.)	Cathedral	Living	Found living in Western Africa.

From what has been said above it is clear that the Karnul fauna represents a stage which preceded the present distribution of the mammals, because we have got here a few species which are entirely extinct, a few that are found now outside India, and a few which are found within India, but in the region to the north of southern India. As remarked by Lydekker, the fauna is decidedly newer than the Narbada gravels and accordingly the cave deposits cannot be older than the upper Pleistocene. It is also younger than the Jumno-Gangetic alluvium and represents the topmost stage of the Pleistocene, if not younger.

The Kashmir caves are at Imselwara and, according to Radcliffe[1], they are two in number and in the smaller of them have been found (*i*) the sambar, (*ii*) *Sus scropha* or the European pig, (*iii*) the teeth of an antelope and (*iv*) the tusk of a bear. No human remains have been found in it as yet and consequently the cave fauna has no value from an anthropological point. Radcliffe thinks

[1] Indian Forester, Vol. 32, pp. 313-'14, 1906.

that these caves are probably of the same age as those of Karnul and this opinion seems to be approved of by Sir Henry Hayden.[1] The known amount of the Imselwara cave fauna is very meagre and it is quite premature to hazard any opinion regarding the age of the cave deposit but, from what has been published about it, the deposit appears to be younger than the Karnul cave fauna and may provisionally be better recognised as sub-recent than Pleistocene.

Conclusion.

My original idea was to give in this paper a review of all the pre-historic finds hitherto recorded in India, but as I proceeded I found that the treatment of such a vast subject would be a stupendous task and accordingly I have confined myself here only to that part of the pre-historic remains by a study of which the age of the man in India can possibly be determined with some accuracy. The deposits containing such materials have been described and, unless further field work is carried out, it will not be possible to say anything more about the age of the artifacts, whether embedded in alluvium or found scattered on the surface, and their relationship to one another. As in Natural History and in Archaeology, so in Pre-history, where a good deal depends upon the collection of materials locally, no true progress is possible unless there are local persons who take a lively interest in the pursuit of these different branches of human knowledge, because, however efficient and enthusiastic a professional worker, whether a Government Official or a University teacher, may be, his visits to the different places of interest must, of necessity, be of a short duration and, if as a result of this discussion, at least one amateur pre-historian comes out, I shall consider all my labours more than amply rewarded.

[1] Rec. Geol. Surv. Ind., Vol. 36, p. 36, 1907.

LIBRARY

505/JOU

24414